Computability Assignment
Year 2012/13 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

1. p(x) = (x is divisible by 4) and q (x) = (x is even) . Then, $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and $P \subset Q$ as P is always smaller than Q.
2. It is not possible . $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ implies that for each element in P, it must also be in Q . So, either P is subset of Q or P is equal to Q.
3. It is not possible. From the argument as explained in 2, P is subset of Q. Hence, P/Q is always empty.
4. The same example as 1. As set Q is always larger than P, $Q \setminus P \neq \emptyset$

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

1. Sufficient to conclude that p_i holds. As A_k is subset of all the sets A_0 to A_{k-1}, all the elements of A_k should belong to all those sets. It can not have any extra element not belonging to all the sets before A_k
2. Sufficient to conclude that p_i holds. Since all the sets A_i are finite, the possible unique subsets of the set is also finite. But there are infinite A_i, hence some of the A_i must repeat. Also, from the condition that $A_{k-1} \supseteq A_k$, only the consecutive sets could be equal. Hence this property is true.

3. Insufficient. If $A_i = \{2, 3, 4, 5\}$, then from the given condition we can say $A_i \supseteq A_{i+1}$which means A_{i+1} can be $\{2, 3, 4, 5\}$ in which case $A_i = A_{i+1}$ is satisfied. But, A_{i+1} can also be $\{2, 3, 4\}$ in which case $A_i = A_{i+1}$ is not true.

4. Insufficient.

5. Sufficient to conclude that p_i holds. The intersection of finite set is always finite.

6. Sufficient to conclude that p_i does not hold. As all sets are infinite and each i+1 th set is subset of i th set, the intersection should be infinite.

7. Sufficient to conclude that p_i holds