Computability Assignment
Year 2012/13 - Number 2

Please keep this file anonymous: do not write your name inside this file.
More information about assignments at http://disi.unitn.it/∼zunino/teaching/computability/assignments

1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

I define this answers:

1. For this case I just find this simple property that satisfy the request : $p(x) := x\, mod\, 4$ and $q(x) := x\, mod\, 2$
2. For this case I just find this simple property that satisfy the request : $p(x) := x\, mod\, 2$ and $q(x) := x\, mod\, 4$
3. For this case I just find this simple property that satisfy the request : $p(x) := x\, mod\, 4$ and $q(x) := x\, mod\, 2$
4. For this case I just find this simple property that satisfy the request : $p(x) := x\, mod\, 2$ and $q(x) := x\, mod\, 4$

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

1. This implication is true because we have that p1 is only a rewriting in a different way of the hypothesis(*).
2. This property is true beacuse to verifiy the hypothesis the chain of sets

   $$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

   is infinite so for maintain the costrain of inclusion $\supseteq$ with set that contain an finite number of element for a certain point we have that the chain of sets contain the same element/s.

3. I can say nothing about this property because this works olny in particular case for example with a set composed of single number or in the case the chain is composed only with the same set.

4. That property is true because we have that in the chain there is no equal set so in this case if we intersect every time somethig “small” at the end we go to have the empty set.

5. If we intersect something that is finite the result is a set that is finite.

6. An intersection of infinite set must generate an infite set so the property is false.

7. Is true for the precedent solution.