Computability Assignment
Year 2012/13 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

Here my answer.

Case 1) Taking p(x):x>0 and q(x):x≥0 we have that the first condition is satisfied for all x belongs to set N. For all the x>0 we have that x both belongs to P and Q, while in case of x=0 we have that x belongs to Q and not to P. We can conclude that P is a subset of Q.

Case 2) The condition can not be satisfied. The sets P and Q need to be equals, so che implication is satisfied when F->F and T->T. Furthermore, the implication need to be true also in case of F->T so thinking to the implication the sets can not to be equals.

Case 3) The condition can not be satisfied. Because taking Taking p(x):x>10 and q(x):x<0 we have that P\Q different from empty but for all the x>10 we have that x belongs to P and not to Q.

the condition of the implication it is not true. The reason is that taking for example x=20, P is true, Q is false so the implication is False.

Case 4) Taking p(x):x<0 and q(x):x>10, and having the sets as P={empty set} and Q={x|x>10} we have that the first condition is satisfied for all x belongs to set N. Regarding the second condition we have Q\P=Q that is different from empty set.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \land i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

Here there is my answer:

Case 1) The hypothesis is not sufficient to conclude that p1 holds. I prove it by a counterexample: taking a generic k equal to 2 we have three sets A0,A1,A2. But it is easy to see that $A2 \neq A0 \bigcap A1 \bigcap A2$, instead A2 is an emprty set.

Case 2) Not answered.

Case 3) The hypothesis is not sufficient to conclude anything about the truth of p3. It depends by which sets we take in consideration. As example, if Ai is empty set, so finite, and $Ai+1=\{1,2,3,4,5\}$, they are finite sets but different.

Case 4) The hypothesis is not sufficient to conclude anything about the truth of p4. For example, a generic set have to be an empty set. If A1 has an element different by another element of A2, it is anyway empty.

Case 5) The hypothesis is sufficient to conclude that p5 holds. The prove is that we have an infinite sequence of sets. Taking two sets Ai, Aj where $i \neq j$. If we compute the intersection of Ai with Aj we have starting form the hypothesis that sets are finite, so we have a finite sets (so called a set $Afinal = Ai \bigcap Aj$). If we iterate this process with all the others sets with index different from i and j, we have again a finite set.

Case 6) and Case 7) The hypothesis is not sufficient to conclude anything about the truth of p6 and p7. It depends by which sets we are taking in consideration. For example having infinite sets but all of them have an common elements, so the intersection in this case is finite. On the other hand we can have many sets that have in common infinite elements and in such case the intersection is an infinite sets.