Computability Assignment
Year 2012/13 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

1. p(x) = x > 4, q(x) = x > 2
2. It is impossible. Assume that there exist some properties p and q such that $x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and $Q \subset P$. This means that there exists some $x \in P$ that is not in Q. Hence it is not true that $x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$, and so we have a contradiction.
3. It is impossible. If $P \setminus Q \neq \emptyset$, that means that there are elements of P which are not in Q, and so it can not be true that $x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$.
4. p(x) = (x mod 6 = 0), q(x) = (x mod 3 = 0)

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

1. True. All the sets from $A_0$to $A_k$contain A_k, so their intersection is A_k.
2. True. If all the sets are finite, and each set is a subset of the previous, from a certain point all the sets will be equal, as they reach the maximum size and so it is not possible to add (nor to remove) elements.
3. the hypothesis is not sufficient to conclude anything about the truth of p_3. As we have seen with p_2, only after a certain i we can be sure that if all the sets are finite $A_i = A_{i+1}$
4. the hypothesis is not sufficient to conclude anything about the truth of p_4. If all the sets are different, it means that there exist some infinite sets, but their intersection has not to be empty.

5. True. The intersection of all the sets can contain at most the number of elements of the smallest set (henceis finite).
6. False. As all the sets are infinite, and each one is a subset of the previous, their intersection will be infinite
7. True. As all the sets are infinite, and each one is a subset of the previous, their intersection will be infinite