Computability Assignment
Year 2013/14 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

1. $P = \{x \in \mathbb{N} | x > 3\}$,$Q = \{x \in \mathbb{N} | x > 1\}$
2. It is not possible to find two properties $p(x)$ and $q(x)$ where $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and $Q \subset P$ hold. By contraddiction, if we assume that there is an element $x \in P \wedge x \notin Q$, then since $x \in P$, it must be that $p(x)$ holds, but since $x \notin Q$ $q(x)$ is false, therefore $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ becomes false.
3. In order for $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ to be true, it must be $P \subset Q$. Otherwise if there is an element of P which doesn't belong to Q we can apply the reasoning at point 2. Therefore it is not possible to find two properties for which the case at point 3 is true.
4. see point 1.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

1. The property holds. Since every set is contained in the ones preceeding it, the intersection of a set A with its predecessors is the set A, therefore the property $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$ is true.
2. The property holds. Since $\forall i \in \mathbb{N}.\ A_i$ is finite and $\forall i \in \mathbb{N}\ A_{i+1} \subset A_i$ it must be that $\forall i \in \mathbb{N} A_{i+1}$has the same number of elements of A_i or less. Morevore, since the sequence of sets is infinite it must be that sooner or later two sets have the same number of element. In fact, if this weren't true, then every set must have less elements than the one preceeding it, but the number of element inside every set is finite and the sequence of sets infinite instead. For this reason from a certein point on, all the sets will have no elements and therfore they would be equal.

3. The hypotesis is not sufficient to conclude anything.
   In order for p_3 to be false, it could be that $A_0$is the set with the first 10 natural numbers and $A_1$the set of the first 5 natural numbers. The other case is trivial instead.

4. The hypothesis is not sufficient to conclude anything.
   If we define the sets $A_0, A_1, A_2...$ as $A_0 = \{10\} \cup \{10 \times i\}, A_1 = \{10\} \cup \{20 \times i\}, A_2 = \{10\} \cup \{30 \times i\}, ...\forall i \in \mathbb{N}$ for example, we obtained that the intersection is egual to the set $\{10\}$ and therefore the property is false.
   If instead we define the set $A_n = \mathbb{N} \backslash \{0, ..., n\}. n \in \mathbb{N}$ we obtain that the property is true, since there is no number we can pick up and that belongs to all the sets $A_i$with $i \in \mathbb{N}$.

5. The property holds. Since the intersection of the complete sequence coincide with the last set and since it is finite, the intersection cannot be infinite.

6. The hypotesis is not sufficient. In fact it can be that $\forall i \in \mathbb{N}, A_i = \mathbb{N}$. In this case the intersection would be $\mathbb{N}$ which is not finite. On the other hand, we could also have the same situation as the one at point 4.

7. The hypotesis is not sufficient for the same reasons at point 6.