Computability Assignment
Year 2012/13 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

1. Let $\forall x \in \mathbb{N}(q(x) \text{ holds})$ and$\forall x \in \mathbb{N}(p(x) \text{ holds} \Leftrightarrow x \text{ is even})$. So P is the set of the even numbers and $Q = \mathbb{N}$. P is strictly included in Q and $\forall x \in \mathbb{N}(p(x) \Rightarrow q(x))$.
2. It's impossible. $Q \subset P \Rightarrow \exists x \in \mathbb{N}(p(x) \wedge \neg q(x))$, but $\forall x \in \mathbb{N}(p(x) \Rightarrow q(x)) \equiv \neg\exists x \in \mathbb{N}\neg(\neg p(x) \vee q(x)) \equiv \neg\exists x \in \mathbb{N}(p(x) \wedge \neg q(x))$.
3. As above, it's impossible. $\forall x \in \mathbb{N}(p(x) \Rightarrow q(x)) \Rightarrow P \subseteq Q$ but $P \setminus Q \neq \emptyset \Rightarrow P \subsetneq Q$.
4. Taking the p and q from the first point is still valid, $\mathbb{N} \setminus P \neq \emptyset$.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

1. $(*)$ is sufficient. $(*) \Rightarrow \forall k \in \mathbb{N}(\forall x(x \in A_{k+1} \Rightarrow (x \in A_k)) \Rightarrow \forall x \in A_k(x \in A_{k-1} \wedge x \in A_{k-2} \wedge\ \ldots\ \wedge x \in A_0)$. So $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$.
2. $(*)$ is sufficient. $(*) \Rightarrow \forall i \in \mathbb{N}(A_i \text{ finite } \Rightarrow |A_{i+1}| \leq |A_i|)$. $\forall i \in \mathbb{N}(A_i \text{ finite }) \Rightarrow A_0$ finite. Size of the sets can only decrease or remain constant. In the first case, there will obviously be an number h such that $A_h = \emptyset$ and for $(*)$ $A_{h+1} \subseteq A_h \Rightarrow A_{h+1} = \emptyset = A_h$. In the second case $|A_i| = |A_{i+1}| \wedge (*) \Rightarrow A_i = A_{i+1}$.
3. We can't conclude anything about p_3. Let $A_0 = \{1, 2\}$, $A_1 = \{2\}$, $\forall i \in \mathbb{N}\backslash\{0, 1\}(A_i = \emptyset)$. $(*)$ is satisfied but p_3 isn't, because $\exists x \in \mathbb{N}(A_i \text{ is finite} \wedge A_i \neq A_{i+1}(i = 0)$. But we can also find a family such that both $(*)$ and p_3 are satisfied, so $(*)$ isn't sufficient to conclude that p_3 does not hold. For instance, $\forall i \in \mathbb{N}(A_i = \{0\})$.

4. We can't conclude anything about p_4. For instance, the family of sets $A_0 = \mathbb{N}$, $\forall i \in \mathbb{N}\setminus\{0\}(A_i = A_{i-1}\setminus\{i\})$ satisfies $(*)$ and $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, but $\bigcap_{i=0}^{\infty} A_i = \{0\}$ so p_4 doesn't hold. But there are also families of sets that satisfies both $(*)$ and p_4. For instance, $A_0 = \mathbb{N}\setminus\{0\}$, $\forall i \in \mathbb{N}\setminus\{0\}(A_i = A_{i-1} \setminus \{i\})$.

5. $(*)$ isn't even necessary to prove that p_5 holds. Given two sets A and B, $|A \cap B| \leq min\{|A|, |B|\} \Rightarrow |A \cap B| \leq |A| \wedge |A \cap B| \leq |B|$. $\forall i \in \mathbb{N}(A_i \text{ is finite}) \Rightarrow \exists k \in \mathbb{N}(k \text{ is finite}) \Rightarrow |\bigcap_{i=0}^{\infty} A_i| \leq |A_k| \Rightarrow \bigcap_{i=0}^{\infty} A_i$ is finite.

6. We can't conclude anything. Let $\forall i \in \mathbb{N}(A_i = \mathbb{N})$, $\forall i \in \mathbb{N}(A_{i+1} = A_i) \Rightarrow \forall i \in \mathbb{N}(A_{i+1} \subseteq A_i)$ and $\bigcap_{i=0}^{\infty} A_i = \mathbb{N}$, so this family satisfies $(*)$ but not p_6. Now consider $A_0 = \mathbb{N} \setminus \{0\}$, $\forall i \in \mathbb{N} \setminus \{0\}(A_i = A_{i-1} \setminus \{i\})$: again $\forall i \in \mathbb{N}(A_{i+1} \subseteq A_i)$ but this time $\bigcap_{i=0}^{\infty} A_i = \emptyset$, so both $(*)$ and $p_6$are satisfied.

7. We can't conclude anything. See the families of sets at point 6.