Computability Assignment
Year 2012/13 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

Answer:

1. p(x) = (x % 4 = 0), q(x) = (x % 2 = 0);
2. If element x satisfies property p, it satisfies property q, which means that it is inserted both in P and in Q, then Q could have other elements that do not satisfy property p, but at the very least, it contains all the elements that verify p, and therefore all the elements in P, which would make it at the very least equal to Q;
3. Same reason given above: if Q contains at the very least exactly the same elements as P, it follows that subtracting those from P would give back the empty set;

4. (x) = (x % 4 = 0), q(x) = (x % 2 = 0).

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite

3.1 Answer

Answers: need to ask about the difference between “the implication doesn’t hold” and “no sufficient information to conclude”, seem to coincide.

1. Hyp sufficient to conclude that it holds, because $\supseteq$is transitive and the k-th A set is the one contained in every other set of the list;

2. Hyp sufficient to conclude that the implication holds, because if the 0-th A contains a finite number of elements, and the others sets are its subsets, and they are an infinite number, in the subsequent sets either the number of elements decreases, but only for a finite number of times (equal to the number of elements contained in the 0-th A set), in which case when reaching zero there will be a repetition (all empty set), or it wont, in which case we will immediately encounter a repetition (since the next set in this case would be a subset with the same number of elements, the two sets are equal);

3. Hyp not sufficient to conclude that the implication holds, because the i+1-th set could be a proper set of the i-th A, and contain one less element;

4. Hyp sufficient to conclude that the implication doesn't hold, since subsequent elements could be not equal, and still have an infinite number of elements (for example, all the natural numbers and all odd natural numbers), thus there is not quarantee that the number of elements in a set will steadily decrease until reaching the empty set (being the number of sets infinite, rather, this would contradict the hypothesis that the i-th and i+1-th sets would not be equal, because when the number of elements in a set would reach zero, all subsequent sets will also be equal to the empty set);

5. Hyp sufficient to conclude that the implication holds, the intersection will contain at most a number of elements equal to that of the 0-th A (specifically, in the case that all the sets are equal);

6. Hyp sufficient to conclude that the implication doesn't hold: since each subset is contained in the previous one, there are no elements in the i+1-th set that are not present in the i-th set (meaning that the intersection won't contain less elements than the number of elements in the i+1-th set, which is infinite);

7. Hyp sufficient to conclude that the implication holds, for the reason given at point 6.