Computability Assignment
Year 2012/13 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

1. If we define that *p(x)* holds when *x is a multiple of 4* and *q(x)* holds when *x is an even number*, we have the set P that contains the multiples of 4 and the set Q that contains the even numbers. In this case we have that both $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and $P \subset Q$ are valid.
2. In this case is impossible to find a valid definition for *p(x)* and *q(x)* that satisfies both $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and $Q \subset P$. The reason is the following:

   $$Q \subset P \Rightarrow \exists x \in \mathbb{N}.\ x \in P \wedge x \notin Q \Rightarrow \exists x \in \mathbb{N}.\ p(x) \wedge \neg q(x) \not\Rightarrow \forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$$

3. Starting from the definition of set-theoretic difference, $P \setminus Q \neq \emptyset \Rightarrow \exists x \in \mathbb{N}.\ x \in P \wedge x \notin Q$. For this reason, whatever is the definition of $p(x)$ and $q(x)$, $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ is not valid.

4. A valid definition for $p(x)$ and $q(x)$ for this case is the same as in case 1 because since we have $P \subset Q$ this implies $Q \setminus P \neq \emptyset$.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i \in \mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \land i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i \in \mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

1. The hypothesis $(*)$ is sufficient to conclude that p_1 holds because of the definition of intersection.

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots \supseteq A_k \Longrightarrow A_k = \bigcap_{i=0}^{k} A_i$$

2. The hypothesis $(*)$ is sufficient to conclude that p_2 holds because, since all the sets are finite and the inclusions are not strict then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$. An example should be a set A_0 with cardinality 3 and since we have $\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3$ then if $A_3 \neq \emptyset$ it means that $\exists j \in \mathbb{N}.\ 0 \leq j \leq 3$ such that $A_j = A_{j+1}$.

3. The hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_3 because, even if all the sets are finite and the inclusions are not strict then it is not a rule that in this situation all the sets are equal. It may be possible that all the sets are equal, but it may be also not possible.

4. The hypothesis $(*)$ is sufficient to conclude that p_4 holds because of the definition of intersection and since in this case all the sets are different then the intersection among them is empty.

5. The hypothesis $(*)$ is sufficient to conclude that p_5 holds because in the worst case the intersection will be empty, so it is finite anyway.

6. The only case in which p_6 holds is when the intersection of all sets is empty (i.e. $\bigcap_{i=0}^{\infty} A_i$ is finite) and this happens when all sets are different. Otherwise the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_6.

7. Here the only case in which p_7 holds is when all the sets are equal (the hypothesis $(*)$ allows this) and, as consequence, the intersection of all sets is infinite. Otherwise the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_7.