Computability Assignment
Year 2012/13 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

Suppose $p(x) = x < 5$ and $q(x) = 1 < x < 5$, so $P = \{0, 1, 2, 3, 4\}$ and $Q = \{2, 3, 4\}$.

1. $P \subset Q$ is false, cardinality of P is bigger then cardinality of Q
2. $Q \subset P$ is true, becouse all elements of Q are included in P
3. $P \setminus Q \neq \emptyset$ is true, becouse 2. is true
4. $Q \setminus P \neq \emptyset$ is false, cardinality of P is bigger then cardinality of Q and by 2. all elements of Q are also belong to P

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

1. It holds by induction, base case is $k = 0$ and $A_0 = A_0 \cap N$ obviously true, inductive step $A_k = A_{k-1} \cap A_k$ but by * we know that $A_k$is a subset of all previous sets including base case.
2. It holds. By the definition all next sets are smaller or equal to previous one. If at least two sets are equal we've done, if all sets are smaller then prevoius one and they are finite at some point we will reach some empty set and next set will be also an empty set, so we've done
3. Nothing, choosing two equal sets it holds, choosing two not equal sets it doesn't holds

4. It holds. By the definition all next sets are smaller or equal to previous one, but $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$means that next set can be only smaller then previous one, so at some point we will reach an empty set.

5. It holds. In case all next sets are smaller to previous one $\bigcap_{i=0}^{\infty} A_i = \emptyset$ by p4, in case all next sets are equal to previous one we will have some common finite set as intersection.

6. False. Intersection is however a smallest set from examinated sets, if all examinated sets are infinite and contained one in other, also the smallest set must to be infinite.

7. It holds. Inverse of p6.