Computability Assignment
Year 2012/13 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

No, it is not possible to find such properties such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and to satisfy the conditions 2) and 3).

2)

$$\begin{aligned}
Q \subset P \Rightarrow & \exists x. x \in P \wedge x \notin Q \\
\Rightarrow & \exists x. (p(x) = True \wedge q(x) = False) \\
\Rightarrow & \exists x. \neg (p(x) \Rightarrow q(x)) \\
\Rightarrow & \neg \forall x. (p(x) \Rightarrow q(x)) \\
& \quad contradiction
\end{aligned}$$

3)

$$P \setminus Q \neq \emptyset \Rightarrow \exists x . x \in P \wedge x \notin Q$$

and, following the same reasoning for the condition 2), it is possible to arrive to a contradiction.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i \in \mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap \{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap \{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i \in \mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

1. The hypothesis (*) is sufficient to conclude that p1 holds.
According to (*) A_k is contained in every set $A_i.i \geq 0 \wedge i < k$ and in $\mathbb{N}$. Therefore every element of A_k is contained in them and not exist element of A_k that not exists in the other named sets because, according to (*), A_k is contained in them.

2. The hypothesis (*) is sufficient to conclude that p2 holds.
The sequence of sets of natural number is infinite. But if all A_i is finite it is necessary that $\exists j \in \mathbb{N}.A_j = A_{j+1}$ otherwise, if every set is strictly contained in the preceding set, the sequence could not be infinite.

3. The hypothesis (*) is not sufficient to conclude anything about the property of p3.
The sequence being infinite, it is necessary that some sets in the sequence are equal, but not necessarily all of them; so it is possible that for every set, the following set is strictly contained.

4. The hypothesis(*) is sufficient to conclude that p4 holds.
If we consider the infinite sequence of strictly contained sets of natural numbers, every set that we add to the sequence the intersection of the preceding sets is reduced by at least a natural number and so happens infinitely times. Thinkink that the resulting intersection could be different from the empty set, means that a natural number couldn't be removed but that is impossible.

5. The hypothesis (*) is sufficient to conclude that p5 holds.
The intersection of an infinite number of finite sets satisfyng the property (*) is the smallest of them and, being every set a finite set, the result is finite.

6. The hypothesis (*) is not sufficient to conclude anything about the truth of p6.;
If every set set is $\mathbb{N}$ then the intersection is infinite; but it could be also finite if the sequence would be $\mathbb{N} \supseteq$\{set of all Prime numbers and 4\}$\supseteq$\{set of all Prime numbers\} and all remaining sets beig = \{set of all Prime Numbers\}

7. The hypothesis (*) is not sufficient to conclude anything about the truth of p7.;
I could use the same reasoning used in the previous point.