Computability Assignment
Year 2012/13 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

1.1.1 Point 1

Let's take

$$p = \{x | x \ mod\ 4 = 0\} \qquad q = \{x | x \ mod\ 2 = 0\}$$

- We can easily prove that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ because every number that is divisible by 4 is also divisible by 2.
- $P \subset Q$ can be simply demonstrated by checking:
  - $\forall x \in P. x \in Q$: this is implied by the already proved$\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$

- $\exists x \in Q. x \notin P$: we can prove this for one element to be valid. One example is 2: $2 \in Q$ but $2 \notin P$

1.1.2 Point 2

If we combine all the requirements in one formula we obtain

$$\forall x \in Q. x \in P \quad \wedge \quad \exists x \in P. x \notin Q \quad \wedge \quad \forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$$

Now, we use the logical formulation instead of set formulation

$$\forall x \in \mathbb{N}.\ q(x) \implies p(x) \quad \wedge \quad \exists x \in \mathbb{N}.\ p(x) \wedge \neg q(x) \quad \wedge \quad \forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$$

if we replace the last implication with the logical equivalent with $\forall x \in \mathbb{N}.\ \neg p(x) \vee q(x) \implies \forall x \in \mathbb{N}.\ \neg(p(x) \wedge \neg q(x))$

This clearly conflicts with the requirement that $\exists x \in \mathbb{N}.\ p(x) \wedge \neg q(x)$, so it is proved that there cannot exist two properties that satisfy both the requirements at once.

1.1.3 Point 3

$P \setminus Q \neq \emptyset$ can be verified only iff P contains an element that is not in Q. But from $\forall x \in \mathbb{N}.\ p(x) \implies q(x)$ we can see that Q must be a superset of P, so P cannot contain an element not in Q. We cannot find a p or q that satisfy both formulas.

1.1.4 Point 4

In this case we can simply use the same p and q used in the point 1.

- We already proved that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ is satisfied.
- To show $Q \setminus P \neq \emptyset$ we can simply find an element in Q that is not in P. One of a such element is 2, so the predicates satisfy both the requirements.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i \in \mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i \in \mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

1. $p_1 : A_k = \bigcap_{i=0}^{k} A_i = \left(\bigcap_{i=0}^{k-1} A_i\right) \cap A_k$. We can show this by induction.

   (a) **Base case**. For $k = 0$, we simply have $A_0 = \mathbb{N} \cap A_0$, which is obviusly true.

   (b) **Inductive step**. By applying the hypotesis we can say $A_k = A_{k-1} \cap A_k$ Since by hypotesis A_k is a subset of all the sets $\{A_i | i < k\}$, this simply results in $A_k = A_k$, which is true.

2. p_2 : Take one $i \in \mathbb{N}$. By hypothesis A_i is finite, and has cardinality c_i. Since $A_{i+1} \subseteq A_i$, A_{i+1} has cardinality $c_{i+1} \leq c_i$. For each element A_i we have that

   (a) Its cardinality $c_i = c_{i+1}$: then there exist $j \in \mathbb{N}$ such that $A_j = A_{j+1}$, and the thesis is proved.

   (b) Its cardinality $c_i > c_{i+1}$: This means that A_{i+1} has less elements that A_i. Now, let's take into consideration that the sequence is infinite, but the elements are finite. This means that if we continue all the way without encountering the case (a), we eventually reach the point where A_j has zero elements. In this case we are forced to have $A_{k+1} = \emptyset$. Since $A_k = A_{k+1}$the thesis is proved.

3. p_3 : This is not true, and can be proved by a simple counterexample. Assume $A_i = \{1, 2, 3\}$, which is finite. Take $A_{i+1} = \{1, 2\}$. $A_i \neq A_{i+1}$even if A_i is finite, so the p_3 is false.

4. p_4 : I don't know what it is. it looks inconclusive, but I cannot find a proof.

5. p_5 : Since $\forall i \in \mathbb{N}.\ A_i$ is finite, all the elements of the intersection $\bigcap_{i=0}^{\infty} A_i$ are finite. The interesction of finite elements is obviously finite.

6. p_6 : If we take $\forall i \in \mathbb{N}.\ A_i = \mathbb{N}$ then the hypothesis $(*)$ is satisfied, but $\bigcap_{i=0}^{\infty} A_i$ is still inifinite. $p_6$does not hold.

7. p_7 : I don't know what it is. it looks inconclusive, but I cannot find a proof.