Computability Assignment
Year 2012/13 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

p(x): x is multiple of 4, q(x) is multiple of 2. Every number multiple of 4 is also multiple of 2, thus the property required holds ($\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$). Obviously, as every element that satisfy p(x) satisfies also q(x), it means that $P \subseteq Q$ and this is also impossible that $Q \subset P$. To prove 1) it is enough to note that using the p(x) and q(x) as defined above, 2 (the number) belongs to Q but not to P. From this, it follows that also 4) holds as Q is "greater" than P. 3) instead is always false as it is impossible to find an elements that belong to P and does not belong to Q.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

1. it holds as the sequence of sets of natural number are in a non strictly contained relation. Applying the intersection between the first k sets is equivalent to find the greatest common subset of element, that due to the contained relation, is A_k.
2. yes because, knowing that we have an infinite sequence of sets and they all have finite cardinality,there will be some sets (nearby) that have the same items. Even removing every time one item to produce the next set, having an infinite sequence of finite sets, we will reach some points that we have an empty set and it is replied to the next sets.

3. We cannot say anything as, if A_i is finite, having an infinite sequence of sets, it is not always the case that $A_i = A_{i+1}$, maybe $A_i \subset A_{i+1}$. Only in the case that $A_o = \emptyset$ the property holds, otherwise it does not hold.

4. Imposing to our infinite sequence of sets to be not "equal" between "neighbors", means to have a infinite sequence of sets$A_0 \supset A_1 \supset A_2 \supset A_3 \supset ....$ As they are all different, the intersection between them will be equal to $\emptyset$.

5. If all the set are finite, means that they share (having as hypothesis a infinite sequence of set) a finite part. So the intersection is finite and the property holds.

6. The property does not hold as we have an infinite sequence of infinite sets. As they are not strictly subset of each other, they share some elements. But the sets are all infinite, this means that the intersection between the set is infinite.

7. As said for property 6) the infinite sequence of infinite sets share a part that is infinite. So this property holds.