Computability Assignment
Year 2012/13 - Number 2

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1 Question

In this exercise, $p(x)$ and $q(x)$ will be two unary properties over natural numbers, and P and Q will denote the sets $P = \{x \in \mathbb{N} : p(x) \text{ holds}\}$ and $Q = \{x \in \mathbb{N} : q(x) \text{ holds}\}$. If possible, for each of the cases below find two properties $p(x)$ and $q(x)$ such that $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ and

1. $P \subset Q$ (strict inclusion);
2. $Q \subset P$ (strict inclusion);
3. $P \setminus Q \neq \emptyset$;
4. $Q \setminus P \neq \emptyset$.

If for some of the above cases it's impossible to find such properties, provide a brief explanation of why is it so.

1.1 Answer

1. Let's take $p(x) = \begin{cases} True & \exists n \in \mathbb{N}.(x = 4n) \\ False & o.w. \end{cases}$ and $q(x) = \begin{cases} True & x\,is\,even \\ False & o.w. \end{cases}$
2. In this case the property doesn't holds because $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ claims that if $p(x)$ holds, $q(x)$ must holds too. But given that $Q \subset P$ (strict inclusion), $\exists x \in P. x \notin Q$ that given the definition of P and Q means that $\exists x \in \mathbb{N}. p(x) \wedge \neg q(x)$ but from that, we know that $\exists x \in \mathbb{N}. \neg\neg(p(x) \wedge \neg q(x))$ that for de morgan's laws leads to $\neg \forall x \in \mathbb{N}. \neg(p(x) \wedge \neg q(x))$, $\neg \forall x \in \mathbb{N}.(\neg p(x) \vee q(x))$, $\neg \forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ that is in contraddiction with $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$.

3. $P \setminus Q \neq \emptyset$ is similar to the previous one but is more general since that it can be that there are some elements in Q that aren't in P, btw it is true if $\exists x \in P.\ x \notin Q$ that implies $\exists x \in \mathbb{N}.\ x \in P \wedge x \notin Q$ that implies $\neg\forall x \in \mathbb{N}.\ \neg x \in P \vee x \in Q$ that implies $\neg\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ that is in contraddiction with $\forall x \in \mathbb{N}.\ p(x) \Rightarrow q(x)$ so is not possible to find any $p(x)$ or $q(x)$ that satisfy this condition.

4. The last one is a case similar to the first one so $Q \setminus P \neq \emptyset$ it can be satisfied in the special case $P \subset Q$. So it can be satisfied by $p(x)$ or $q(x)$ defined as in (1.). An interesting property of this is that in contrast with (1.) it is unsatisfiable in a general case in fact it's not mandatory that $\forall x \in P.\ x \in Q$ in fact it can be that $\exists x \in P.\ x \notin Q$ and that's the case of $P = \{x | x\ is\ prime\}$ and $Q = \{x | x\ is\ odd\}$ being prime in fact doesn't implies being odd because that $2 \in P \wedge 2 \notin Q$.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcap_{i=0}^{\infty} A_i = \bigcap\{A_i \mid i \in \mathbb{N}\} = \{x \mid \forall i \in \mathbb{N}\ x \in A_i\}$ and $\bigcap_{i=0}^{k} A_i = \bigcap\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cap A_1 \cap \cdots \cap A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$\mathbb{N} \supseteq A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \cdots (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcap_{i=0}^{k} A_i$;
2. p_2: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then there exists $j \in \mathbb{N}$ such that $A_j = A_{j+1}$;
3. p_3: for all i, if A_i is finite, then $A_i = A_{i+1}$;
4. p_4: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcap_{i=0}^{\infty} A_i = \emptyset$;
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is finite;
7. p_7: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcap_{i=0}^{\infty} A_i$ is infinite.

3.1 Answer

1. the hypothesis $(*)$ is sufficient to conclude that p_1 holds. In fact for any set A_k we have that $A_k \subseteq A_{k-1} \subseteq ... \subseteq \mathbb{N}$ that means that for the definition of inclusion and by our assumption p_1, $\forall x \in A_k.x \in A_{k-1} \wedge ... \wedge x \in A_0 \wedge x \in \mathbb{N}$. now let's suppose by contraddiction that $\exists k \in \mathbb{N}.\ A_k \neq \bigcap_{i=0}^{k} A_i$ from that we can derive that either $\exists x \in \bigcap_{i=0}^{k} A_i.x \notin A_k$ or $\exists x \in A_k.x \notin \bigcap_{i=0}^{k} A_i$. The first alternative is in contrast with the hypothesis in fact doesn't exists any set A_i such that $i < k$ where $A_i \subset A_k$. The second one is in contraddiction with $\forall x \in A_k.\ x \in A_{k-1} \wedge ... \wedge x \in A_0 \wedge x \in \mathbb{N}$ so we need to distcard $\exists k \in \mathbb{N}.\ A_k \neq \bigcap_{i=0}^{k} A_i$ and accept our assumption p_1.

2. the hypothesis $(*)$ is sufficient to conclude that p_2 holds. If $\neg\exists j \in \mathbb{N}. A_j = A_{j+1}$ so $\forall j \in \mathbb{N}.\ A_j \neq A_{j+1}$ this assumption leads us to restrict the hypothesis to $\mathbb{N} \supset A_0 \supset A_1 \supset A_2 \supset A_3 \cdots$ in this case for the fact that all the sets A_i are finite $\exists j \in \mathbb{N}. A_j = \emptyset$ but from the restricted hypothesis we have that $A_j \supset A_{j+1}$ that means that $\exists x \in \mathbb{N}.\ x \in A_j \wedge x \notin A_{j+1}$, for $A_j = \emptyset$ we have $\exists x \in \mathbb{N}.\ False \wedge x \notin A_{j+1}$ that makes it false no matter what, so we reached a contraddiction. We need to discard the assumption $\neg\exists j \in \mathbb{N}.\ A_j = A_{j+1}$ and, because of that, we accept our assumption p_2.

3. the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_3. In fact p_3 can be either True or False given the hypothesis. Assuming that the property holds we have that $\mathbb{N} \supseteq A_0 = A_1 = ...$ that it's not in contraddiction with the hypothesis. Now assuming that the property doesn't holds we have that $\exists i \in \mathbb{N}.\ A_i \neq A_{j+1}$ that means that either $A_i \subset A_{i+1}$ or $A_i \supset A_{i+1}$ must be true, the first is discarded by the hypothesis but the second case is in line with the hypothesis. So given the hypothesis we can't say anything about the truth of p_3

4. the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_4 . If we assume that $\forall i \in \mathbb{N}. A_i\ is\ finite$ then p_4 is false because that p_2 holds. If indeed we assume that for all i, A_i is infinite we can prove that p_4 holds. Let's A_0 be an infinite set, for the hypothesis and p_4 we have that $A_1 \subset A_0$ so there must $\exists x \in A_0.\ x \notin A_1$. let's call this x_0. this can be done for any generic $n \in \mathbb{N}$, $A_n \setminus A_{n+1} = \{x_n\}$. Now let's suppose by contradiction that $\bigcap_{i=0}^{\infty} A_i \neq \emptyset$ so there $\exists k \in \mathbb{N}.\ x_k \in \bigcap_{i=0}^{\infty} A_i$ but this is a contraddiction because the element $x_k \notin A_{k+1}$ so $\bigcap_{i=0}^{\infty} A_i = \emptyset$.

5. the hypothesis $(*)$ is sufficient to conclude that p_5 holds; If $\forall i \in \mathbb{N}.\ A_i$ is finite then $\bigcap_{i=0}^{\infty} A_i$ is finite. we have that $\forall i \in \mathbb{N}.\exists k \in \mathbb{N}.\ |A_i| = k$, since that we know that A_i is finite we can enumerate each element in it form $x_0, ..., x_k$ for $k = |A_i|$ and that the smallest set into the series $|A_{min}| = min$. Now let's assume that $\bigcap_{i=0}^{\infty} A_i$ is infinite this means that $x_{min+1} \in \bigcap_{i=0}^{\infty} A_i$ too, but this is in contraddiction with the definition of intersection because $x_{min+1} \notin A_{min}$ that's is one of the sets into the series so we discard $\bigcap_{i=0}^{\infty} A_i$ is infinite and accept p_5

6. 7. the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_6 and p_7. In fact if $\mathbb{N} \supseteq A_0 = A_1 = A_2 = A_3 = ...$, the intersection of infinite set $\bigcap_{i=0}^{\infty} A_i = A_0 = A_1 = A_2 = A_3 = ...$ thats is infinite, but if for instance $\mathbb{N} \supseteq A_0 \supset A_1 \supset A_2 \supset A_3 \supset ...$ we have already proved in the proof 4. that it's finite and exactly $\bigcap_{i=0}^{\infty} A_i = \emptyset$ so we can say anything about the property.