Computability Assignment
Year 2012/13 - Number 7

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1 Question

Prove that the following set is not λ-definable.

$$A = \{\#M \mid M \text{ has a } \beta\text{-normal form}\}$$

(Hint: show that, if A were λ-definable, then also K_λ would be λ-definable, hence obtaining a contradiction.)

1.1 Answer

Assume that we can define a verifier for A, called V_A, that works in this way:

$$V_A \ulcorner n \urcorner = \begin{cases} T & if\ n \in A \\ F & otherwise \end{cases}$$

In particular, the program V_A takes in input a number (that is the encoding of a program) and tells if it has or not a β- normal form. A program M can be in the form $N \ulcorner N \urcorner$ and, in this case, A contains $\#(N \ulcorner N \urcorner)$. If we use this idea we can use V_A to create a validator for K_λ. So, given in input a number (the encoding of a N) we check if in A exists the encoding of $N \ulcorner N \urcorner$: if yes, then the encoding of N belongs to K_λ, otherwise no.

$$V_{\mathsf{K}_\lambda} = \lambda n. V_A (App\ n\ (Num\ n))$$

With this, K_λ is λ-definable but we know that this is impossible so we get a contradiction.

2 Question

Let A be a λ-definable set. Prove that

$$B = (A \cup \{b_1, \ldots, b_n\}) \setminus \{c_1, \cdots, c_m\}$$

is also λ-definable.

(Hint: do not reinvent the results we saw in class, just apply them.)

2.1 Answer

First, if A is λ-definable, we can assume that the verifier V_A is valid for the set A. After that we can construct the verifiers of finite sets $D = \{b_1, ..., b_n\}$ and $C = \{c_1, ..., c_m\}$ as we have done in class, so by simply doing all possible checks on the element passed:

$V_D = \lambda x.Equal\ x \ulcorner b_1 \urcorner\ T\ (Eq\ x \ulcorner b_2 \urcorner\ T\ (...(Eq\ x \ulcorner b_n \urcorner\ T\ F))$
$V_C = \lambda x.Equal\ x \ulcorner c_1 \urcorner\ T\ (Eq\ x \ulcorner c_2 \urcorner\ T\ (...(Eq\ x \ulcorner c_n \urcorner\ T\ F))$

Now we can build a verifier for B, called V_B that defines a valid verifier

$V_B = \lambda x.And\ (Or\ (V_A\ x)\ (V_D\ x))\ (Not\ (V_C\ x))$

3 Question

Let A be a **non** λ-definable set. Prove that

$$B = (A \cup \{b_1, \ldots, b_n\}) \setminus \{c_1, \cdots, c_m\}$$

is also **non** λ-definable.

(Hint: prove the contrapositive. That is, prove that if B were λ-definable, then also A would be such.)

3.1 Answer

We can prove that by logica implications. Take the verifiers from the previous exercise, so that V_D and V_C are valid verifiers. We know, in this case, that the definability of the set B is linked to the one of the set A, in particular we have that $A\ is\ \lambda - definable \Rightarrow B\ is\ \lambda - definable$. So, this implication can be seen on the other side, in particular $B\ is\ not\ \lambda - definable \Rightarrow A\ is\ not\ \lambda - definable$.

By the exercise we have that A is not λ-definable

(RZ: so far so good)

and so must be also B, because it is the only way in which this last implication is true.

(RZ: absolutely not! you are stating that from $p \implies q$ and q one can deduce p, which is just wrong! Be careful!)