Computability Assignment
Year 2012/13 - Number 4

Please keep this file anonymous: do not write your name inside this file.

More information about assignments at http://disi.unitn.it/~zunino/teaching/computability/assignments

Please do not submit a file containing only the answers; edit this file, instead, filling the answer sections.

1 Preliminaries

A partial function g is said to be a *restriction* of a partial function f , written $g \subseteq f$ iff

$$\forall x \in \mathsf{dom}(g).\ g(x) = f(x)$$

Note: this notation "overloads" the symbol $\subseteq$. Indeed, we shall write $A \subseteq B$ to express a subset relation between two sets, and $g \subseteq f$ to express a restriction relation between two functions.

(From a formal point of view, since we defined functions as set of pairs the two notions coincide: the restriction relation above is equivalent to requiring that $\langle a, b\rangle \in g \implies \langle a, b\rangle \in f$ for all a, b, which indeed states that g is a "subset" of f).

2 Question

Let $\mathcal{F}$ be the set of partial functions $\{f \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | \forall x \in \mathbb{N}.\ f(2 \cdot x) = x\}$.

- Define two distinct partial functions f_1, f_2 which belong to $\mathcal{F}$. (I.e, provide two such examples.)
- Define two distinct partial functions g_1, g_2 which do *not* belong to $\mathcal{F}$. (I.e, provide two such examples.)
- Define a partial function $f \in \mathcal{F}$, and consider the set of its *finite* restrictions $\mathcal{G} = \{g \in (\mathbb{N} \rightsquigarrow \mathbb{N}) | g \subseteq f \ \wedge\ \mathsf{dom}(g) \text{ finite}\}$.
  - Define two distinct partial functions h_1, h_2 which belong to $\mathcal{G}$. (I.e, provide two such examples.)

– Prove whether $\mathcal{F} \cap \mathcal{G} = \emptyset$.

2.1 Answer

The requirements are that the fuctions are partial from the set of natural number to the set of natural number and that, on even numbers, they returns one half the value.

- We can define two partial functions as follows. The first is $f_1(x) = \begin{cases} x/2 & if\, x\, is\, even \\ undefined & otherwise \end{cases}$, so if the value in the domain in even it returns the half and the function is partial. The second one is $f_2(x) = \begin{cases} x/2 & if\, x\, is\, even \\ x & otherwise \end{cases}$. In this case we do not have undefined values but it is still a partial function by definition and the property $f(2 \cdot x) = x$ is preserved.

- We can define two patial functions $g_1, g_2 \in (\mathbb{N} \rightsquigarrow \mathbb{N})$ that do not belog to $\mathcal{F}$ as follows. The first can be $g_1(x) = \begin{cases} x & if\, x < 10 \\ undefined & otherwise \end{cases}$ and the second can be $g_2(x) = \begin{cases} 2 \cdot x & if\, x > 5 \\ undefined & otherwise \end{cases}$. They basically do not belong to $\mathcal{F}$ because they do not satisfies the condition expressed by $f(2 \cdot x) = x$.

- We can take the definition of f_1 and call it just f. So, h_1, h_2 must be restrictions of this f and they must have a finite domain that is, in the case of a partial function, the set of value for which it is defined. So these functions are: $h_1(x) = f(x), \forall x < 10$ and $h_2(x) = f(x), \forall 10 < x < 100$. Basically we make the domain finite by restricting the domain of f to some values, in particular in the first case to the ones that are less then 10 and the second to the one between 10 and 100. To prove that $\mathcal{F} \cap \mathcal{G} = \emptyset$ we just have to analyze the domain of the functions. In the set $\mathcal{F}$ we have only functions with infinite domain because they must be definite on all even values of $\mathbb{N}$ while in the $\mathcal{G}$ set we have only functions that have finite domain by definition. Given that a function cannot have finite and infinite domain at the same time, the intersection between these two sets will always be empty.

Note.

The next part is an advanced exercise. I'd suggest to **skip** it, unless you want an extra challenge.

3 Preliminaries

Let $\mathcal{R}$ be a set of inference rules over elements of a set A. Then, $\mathcal{R}$ induces a function $\hat{\mathcal{R}} \in (\mathcal{P}(A) \to \mathcal{P}(A))$ given by

$$\hat{\mathcal{R}}(X) = \{y \mid \exists(\frac{x_1 \dots x_n}{z}) \in \mathcal{R} \land y = z \land \forall i \in \{1, \dots, n\}.x_i \in X\}$$

4 Question

Let m, n range over natural numbers. Consider the following set of inference rules $\mathcal{R}$

$$\frac{n \ m}{n \cdot m} \qquad \frac{}{1} \qquad \frac{n}{n \cdot 2}$$

an the sets

$$E = \{2 \cdot n \mid n \in \mathbb{N}\} \qquad O = \{2 \cdot n + 1 \mid n \in \mathbb{N}\}$$

Then, answer the questions below.

1. State whether $\hat{\mathcal{R}}(O) \subseteq O$
2. State whether $O \subseteq \hat{\mathcal{R}}(O)$
3. State whether $\hat{\mathcal{R}}(E) \subseteq E$
4. State whether $E \subseteq \hat{\mathcal{R}}(E)$
5. State whether $\hat{\mathcal{R}}(\mathbb{N}) \subseteq \mathbb{N}$
6. State whether $\mathbb{N} \subseteq \hat{\mathcal{R}}(\mathbb{N})$
7. State whether $\hat{\mathcal{R}}(E \cup \{1\}) \subseteq E \cup \{1\}$

You may whish to exploit the answer for some question when answering another. Finally:

1. Characterize the minimum fixed point of $\hat{\mathcal{R}}$, i.e. $\bigcap\{X \mid \hat{\mathcal{R}}(X) = X\}$
2. Characterize the maximum fixed point of $\hat{\mathcal{R}}$, i.e. $\bigcup\{X \mid \hat{\mathcal{R}}(X) = X\}$

4.1 Answer

Notice that set E represent the set of even number while the set O represent the set of odd number.

1. If we can apply the rules just one time we find that the set $\hat{\mathcal{R}}(O)$ contains all the odd number (found by applying the first rule with 1 as n and inclunding 1 by the second rule) and all the double of every odd number, due to the last rule. Si the set $\hat{\mathcal{R}}(O)$ is bigger that the set O and by that it is not a subset and so the property is false.

2. In this case, as I explained in the previous point, the condition is satisfied because $\hat{\mathcal{R}}(O)$ is just O unite with the set of double of all the odd numbers and so the property is true.

3. The set $\hat{\mathcal{R}}(E)$ contains all the even numbers (that can be found with the last rule) except for the ones that are calculated from a odd number (like 2, 6, 10 and so on) and plus the element 1 (found with the second rule). In particular, due to this last case, it is not a subset of the even numbers set E and so the property is false.

4. For the reasons explained in the previous point, in particular due to the fact that $\hat{\mathcal{R}}(E)$ does not contain every even number, the property is false.

5. By simply applying the first rule with $n = 1$ we can found all the numbers in $\mathbb{N}$ except for 1 (RZ: actually, 1 can be generated by $n = m = 1$) that we get from the second rule. By that, $\hat{\mathcal{R}}(\mathbb{N})$ is the set of all natural numbers $\mathbb{N}$ and so the property is true.

6. Like before, the two sets are the same so even this time the property is true.

7. With the inclusion of the number 1 we can find (by applying the first rule with 1) all the even numbers. So, if in 3. and 4. some even numbers were missing, now we can have all of them and by that the two sets are equal so the property is true.

Last two points:

1. In order to find the minumum fixed point we need to start from the emtpy set and make steps by applying the $\hat{\mathcal{R}}$ function and until we reach a stable solution. In particular this is achieved by the set of power of two. At the first step we add just one, then by applying the function to this set we get $X = \{1, 2\}$ (due to the first or the last rule), then we add also 4 (due to the same two rules) and so on.

2. The maximum fixed point, as we saw in previous exercise, should be just $\mathbb{N}$.