Computability Assignment
Year 2012/13 - Number 3

Please keep this file anonymous: do not write your name inside this file.

More information about assignments at http://disi.unitn.it/∼zunino/teaching/computability/assignments

Please do not submit a file containing only the answers; edit this file, instead, filling the answer sections.

1 Question

Let A, B be sets, and let $\mathsf{id}_A, \mathsf{id}_B$ denote the identity functions over A and B respectively. Assume $f \in (A \to B)$ and $g \in (B \to A)$ be functions satisfying $g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$. Prove that f is a bijection (i.e., injective and surjective).

1.1 Answer

We can use the two assumptions to prove that f is a bijection; in particular the first one will be used to prove the injectivity property (1.) and the second one to prove the surcjectivity property (2.).

1. We define the injectivity as $\forall x, y \in dom(f).f(x) = f(y) \Rightarrow x = y$ and assume that f is not injective. So if we apply f to two elements $x, y \in A$, even if $f(x) = f(y)$ they can be different elements that leads to the same value by applying the function. Because of that, when we try to apply the g function, we can get a different element from the one passed to f and this is not an identity function. In particular we can have $(g \circ f)(x) = y$ and$(g \circ f)(y) = x$.
2. We define the surjectivity as $\forall b \in B.\exists a \in A.f(a) = b$ or, alternatively, if the codomain is equal to the image and assume that f is not surjective. We define$b \in B.b \notin img(f)$. So, the g function can be applied to element b and we get an element $a \in A$. But, if we try to apply the function f to this element, we cannot get b because it is not part of the image of f and so the identity function is not equal to $f \circ g$.

By these, the function must be injective and surjective and so bijective.

2 Question

Let A, B be sets, and let $f \in (A \leftrightarrow B)$ be a bijection. Define a bijection $g \in (\mathcal{P}(A) \leftrightarrow \mathcal{P}(B))$ and prove it is such.

2.1 Answer

The idea is that if exists a bijection between the two sets, called f, we can group elements of A (more precisely they are $\mathcal{P}(A)$) and do the same with the corresponding elements in B, where the match is given by the function f. So we can define $X \in \mathcal{P}(A)$ and the function $g(X) = \{y \in B \mid \forall x \in X.f(x) = y\}$. (RZ: maybe $g(X) = \{y \in B \mid \exists x \in X.f(x) = y\}$) Now we can prove the bijection (1. for injection and 2. for surjection):

1. Assume that g is not injective and take $Y = g(X) = g(X')$ but $X \neq X'$. By that and by the definition of g, $\exists x \in X, x' \in X'.f(x) = f(x') = y$ with $x \neq x'$ but in this case we have a contradiction because f must be injective.
2. Assume that g is not surjective. So by definition of g we have that can exist an element y that cannot be reached by applying f to a general x. But this is a contradiction because, by hypothesis, f must be surjective.

So, in general, if we use the function f all elements are connected and not exists an element is B that cannot be found applying f (because, in this case, the initial bijection is not preserved) so the injection and surjection is ensured.

3 Question

Let A, B be two sets, and let $b \notin B$. Define a bijection f between the set of partial functions $(A \rightsquigarrow B)$ and the set of total functions $(A \to B \cup \{b\})$. Prove that is is such.

3.1 Answer

We have to find a function that can "transform" a partial function to a total function. The inclusion of the b element allow us to map all the non-defined elements in the given function to it. So f is a function that take in input a function and returns its transformation: take $g \in (A \rightsquigarrow B)$, so $f(g) = g'$ where g' is defined as $g'(x) = \begin{cases} g(x) & if\ g(x)\ is\ defined \\ b & otherwise \end{cases}$. Now we can prove the bijection (1. for injection and 2. for surjection):

1. Assume it is not injective. So we can have that $f(g) = f(h) = i$ but $g \neq h$. In this case $i(x) = g(x)$ and $i(x) = h(x)$ but there must exists a case in which $g(x)$ is different from $h(x)$ (by assumption) and so the two i cannot be equal.

2. Assume it is not surjective. So we can have a function $g \in (A \to B \cup \{b\})$ such that it cannot be mapped from any function in $(A \rightsquigarrow B)$. But always exists an element in the total functions set that can be found from the partial function set, because take a total function; if $\forall x \in A.g(x) = b$ we set the function as undefined, we create a new function that map some elements (subset) of the set A to elements of the set $B \backslash \{b\}$, that is in particular $(A \rightsquigarrow B)$.

Note.

The exercises below are harder. Feel free to skip them if you find them too hard.

4 Question

Define a bijection $f \in [(\mathcal{P}(A) \times \mathcal{P}(B)) \leftrightarrow \mathcal{P}(A \uplus B)]$. Prove that is is such.

4.1 Answer

The product of a subset of A with a subset of B must be in bijection with a subset of the disjoin union of the two sets. We can define the bijective function as $f(\langle X, Y \rangle) = X \biguplus Y$ with $X \subseteq A, Y \subseteq B$. A requirement is that the values of the disjoin union must be constructed in a way that we can find the set they belong. This is simply done by using the same method that we saw in class, so by adding to the value the name of the set (or a value that identify it). With this we can eventually "go back" and retrieve the pair from the disjoint union set. Proof that it is bijective (1. for injective, 2. for surjective):

1. Assume it is not injective. So exists two pairs $\langle X, Y \rangle, \langle W, Z \rangle$ with $X, W \subseteq A$, $Y, Z \subseteq B$ such that $f(\langle X, Y \rangle) = f(\langle W, Z \rangle)$ but $\langle X, Y \rangle \neq \langle W, Z \rangle$ (for some elements, the two pairs are differents). The way in which we defined the elements in the disjoin union allow us to know from which set they are took. For instance we ensure that the two pairs $\langle \{0\}, \{1\} \rangle, \langle \{1\}, \{0\} \rangle$ do not lead to the same union. By that, $X \biguplus Y$ and $W \biguplus Z$ cannot be equal as long as $X \neq W$ or $Y \neq Z$, so this is in contradiction with the assumptions.
2. Assume it is not surjective. So exists a set U that has no corresponding value in the pairs set. But, if we construct two sets as $X = \{x \mid \langle 0, x \rangle \in U\}$ and $Y = \{y \mid \langle 1, y \rangle \in U\}$ (assuming that we identified with 0 the elements of the set A and with 1 the elements of the set B) then we have that $X \subseteq A, X \in \mathcal{P}(A)$ and $Y \subseteq B, Y \in \mathcal{P}(B)$. So we can find a correspondant set in the pair set and this is a contradiction.

5 Question

Define a bijection $f \in [((A \uplus B) \to C) \leftrightarrow ((A \to C) \times (B \to C))]$. Prove that is is such.

5.1 Answer

Define $g : (A \uplus B) \to C)$. We can "force" g to use an element in a particular set instead of an element in the disjoin union set. So we can define $f(g) = \langle h, i \rangle$ where $h(x) = g(\langle 0, x \rangle)$, $i(x) = g(\langle 1, x \rangle)$ and, like before, we assume that we identify the set A with value 0 and the set B with value 1 and the elements in the disjoin union are constructed as always. Proof that it is bijective (1. for injective, 2. for surjective):

1. Assume it is not injective. So exists two functions g, h such that $f(g) = f(h)$ but with $g \neq h$. If we consider the definition of the pair of function that f returns, we can easly see that all the value computed by them are based on the definition of g and h and so they can lead to the same results only if they are equal but, for the assumption they are not. This is a contradiction.
2. Assume it is not surjective. So exists a pair of functions that cannot be reached by any element in the set $(A \uplus B) \to C)$. Take$\langle h, i \rangle$as this particular element. We can construct a function such that $g(\langle x, y \rangle) = \begin{cases} h(y) & if\, x = 0 \\ i(y) & otherwise \end{cases}$.
This new function it can always be created if we have the pair$\langle h, i \rangle$and it is indeed an element of the set $(A \uplus B) \to C)$ so this is a contradiction with the assumptions.

6 Question

Define a bijection $f \in [((A \to (B \times C)) \leftrightarrow ((A \to B) \times (A \to C))]$. Prove that is is such.

6.1 Answer

Similarly as what we have done before we can set $g : (A \to (B \times C))$ and define $f(g) = \langle h, i \rangle$where $h(a) = b$ and $i(a) = c$ with $g(a) = \langle b, c \rangle$. Proof that it is bijective (1. for injective, 2. for surjective):

1. Assume it is not injective. Then exists two functions g, h such that $f(g) = f(h)$ but with $g \neq h$. By the definition of f, all the functions that are computed with g and h, are based on these two functions. So, if g and h are different, the two pair of functions returned by f will produce different values on some particular input and this is a contradiction with the assumption.
2. Assume it is not surjective. Then exist a pair of functions$\langle h, i \rangle$ such that

it does not have a correspondent element in the set $(A \to (B \times C))$. Take $g(a) = \langle b, c\rangle$with $b = h(a)$ and $c = i(a)$. As we can see $g : (A \to (B \times C))$ and it always can be created so this element will always exists and that is in contradiction with the assumption.