Computability Assignment
Year 2012/13 - Number 2

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1 Question

Let A, B be two sets. Prove that the properties below are equivalent.

- $A = \emptyset \vee B = \emptyset (1)$
- $A \times B = \emptyset (2)$

1.1 Answer

We prove that (1) $\implies$ (2), then (2) $\implies$ (1). So we have (1) $\iff$(2).

- From (1), if at least one of two sets are empty, then we couldn't find a $x \in A$ or $y \in B$ or both $x \in A$ and $y \in B$ for an ordered pair $<$x,y$>$ to be existed. Hence from the definition of $A \times B$, we have (2).
- On the contrary, (2), from the definition of $A \times B$, we can't find an ordered pair $<$x,y$>$ such that $x \in A \wedge y \in B$, this implicates that $\nexists x.x \in A$ or $\nexists y.y \in B$ or $\nexists x.x \in A \wedge \nexists y.y \in B$. Then, we have (1).

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i \in \mathbb{N}}$, we define $\bigcup_{i=0}^{\infty} A_i = \bigcup \{A_i | i \in \mathbb{N}\}$ and $\bigcup_{i=0}^{k} A_i = \bigcup \{A_i | i \in \mathbb{N} \wedge i \leq k\} = A_0 \cup A_1 \cup \cdots \cup A_k$.

3 Question

Assume $(A_i)_{i \in \mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N} (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}. A_k = \bigcup_{i=0}^{k} A_i$
2. p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$
3. p_3: if $\forall i \in \mathbb{N}. A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$
4. p_4: if $\forall i \in \mathbb{N}. A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite
5. p_5: if $\forall i \in \mathbb{N}. A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite
6. p_6: if $\forall i \in \mathbb{N}. A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

3.1 Answer

1. From the left side of p1: $\forall k \in \mathbb{N}$, we can choose $x \in A_k$. We show that x also belongs to the right side. From $\bigcup_{i=0}^{k} A_i = A_o \cup A_1 \cup ... \cup A_k$ and $(*)$, it is easy to see x also belongs to A_k, that is a subset of the right side. So $(*)$ is sufficient to conclude that p_1 holds.

2. We give two examples to prove that if all the conditions are satisfied, we have $A_i \neq A_{i+1}$and also have $A_i = A_{i+1}$.

- Choosing $A_0 = \{1, 2, 3, ...\}, A_1 = \{0, 1, 2, 3, ...\}, A_2 = \{0, 1, 2, 3, ...\}, ...$ The example satisfies $(*)$ and $\forall i. A_i$is infinite . But $A_0 \neq A_1$.
- Choosing $A_0 = \{0, 1, 2, 3, ...\}, A_1 = \{0, 1, 2, 3, ...\}, A_2 = \{0, 1, 2, 3, ...\}, ...$The example satisfies $(*)$ and $\forall i. A_i$is infinite . For all i, $A_i = A_{i+1}$.

  So the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_2.

3. From the condition $\forall i \in \mathbb{N}. A_i \neq A_{i+1}$and $(*)$, we do like p_2.

- We can choose $A_0 = \{2, 3\}, A_1 = \{1, 2, 3\}, A_2 = \{1, 2, 3, 4\}, A_3 = \{1, 2, 3, 4, 5\}, ...$ We have $\bigcup_{i=0}^{\infty} A_i = \{1, 2, 3, ...\} \neq \mathbb{N}$. Because the left-side set doesn't include 0.
- We can choose $A_0 = \{0\}, A_1 = \{0, 1\}, A_2 = \{0, 1, 2\}, A_3 = \{0, 1, 2, 3\}, ...$Then $\bigcup_{i=0}^{\infty} A_i = \{0, 1, 2, 3, ...\} = \mathbb{N}$. So $(*)$ is not sufficient to conclude anything about the truth of p_3.

  4. $\forall i \in \mathbb{N}. A_i$ is finite. There always exists $k \in \mathbb{N}$such that $\forall i \in \mathbb{N}. |A_i| \leq k$. Then with $(*)$, we have$|\bigcup_{i=0}^{\infty} A_i| \leq k$ and it is finite. So $(*)$ is sufficient to conclude that p_4 holds.

5. This is the contrary of p_4. So $(*)$is sufficient to conclude that p_5 does not hold.

6. From $\forall i \in \mathbb{N}.A_i$ is infinite, it is easy to see $\bigcup_{i=0}^{\infty} A_i$ is infinite.So $(*)$is sufficient to conclude that p_6 holds.(It is not necessary to include $(*)$ in this case)