Computability Assignment
Year 2012/13 - Number 2

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1 Question

Let A, B be two sets. Prove that the properties below are equivalent.

- $A = \emptyset \vee B = \emptyset$
- $A \times B = \emptyset$

1.1 Answer

To prove this we have two possible cases:

1. Assume the first property to be true. By definition, the cartesian product between two sets is $A \times B = \{(a, b) \mid a \in A \text{and} b \in B\}$. So, if one of the two sets (or both) is empty, there are no pairs between their elements and the product is an empty set as well.

2. Assume the second property to be true. By the same definition, if the two sets are both not empty (they have at least one element each), we can define pairs between elements. The only possible way to get an empty product set is to have at least one of the two sets empy and this condition is expessed in the first property.

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcup_{i=0}^{\infty} A_i = \bigcup\{A_i \mid i \in \mathbb{N}\}$ and $\bigcup_{i=0}^{k} A_i = \bigcup\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cup A_1 \cup \cdots \cup A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N} \ (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$
2. p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$
3. p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$
4. p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

3.1 Answer

1. By definition, $A_k = A_0 \cup A_1 \cup \ldots \cup A_k$. So the hypothesis $(*)$ is sufficient to conclude that this property must be true because if A_k contains all the previous sets, at the end we have that $A_k = A_k$.

2. The hypothesis does not give any hint on the truthness of p_i (if hypothesis is true, the property can be either true or false). We can use two different cases. In the first just take $\forall i \in \mathbb{N}$, $A_i = \mathbb{N}$. The hypothesis is true and the property is satisfied. In the second case take $A_k = \mathbb{N} \setminus \{0\}$ and $A_{k+1} = \mathbb{N}$ for a $k \in \mathbb{N}$ and such that all the sets satisfies $(*)$ (for example, before set$k$ they are equal to set$k$ and after set$k+1$ they are equal to set$k+1$). The hypothesis is still valid and they are infinite but they are not equal, in particular when the set k is relationed with the next set $k+1$.

3. Even in this case, the hypothesis does not give us information about the property. Consider two cases. In the first we consider $A_0 = \{0\}$and $\forall i \in \mathbb{N}$, $A_i = A_{i-1} \cup \{i\}$ (intuitively A_k contains all natural number less or equal to k) so if we assume $(*)$ than the property holds because the union at the end will be equal to $\mathbb{N}$ and all the sets are different. But, in the second case, take $A_0 = \{1\}$and $\forall i \in \mathbb{N}$, $A_i = A_{i-1} \cup \{i+1\}$ (intuitively,

every set is like the previous one but with a new element and every set do not contains zero). So all the sets are different and even if the condition(∗) is verified, the union of them will not be equal to $\mathbb{N}$ (because it contains also zero).

4. In general, the union of infinite finite sets is not finite (for example if we define$\forall i \in \mathbb{N}$, $A_i = \{i\}$, the union will be just $\mathbb{N}$), but, in this case due to the (∗) condition, we have that every set is contained into the next set. By this we have that the union will tend to be equal to the last (RZ: there is no last element of an infinite sequence) of the $A_i$sets that is finite for the definition so the hypothesis is sufficient to state that the property holds.

5. For the reasons explained in the previous point, the hypothesis implies that the property does not hold.

6. This property is always satisfied because the union of sets that are infinite (we need just one of them) is infinite so the hypothesis does not really help and the property is always true.