Computability Assignment
Year 2012/13 - Number 2

Please keep this file anonymous: do not write your name inside this file.
More information about assignments at http://disi.unitn.it/~zunino/teaching/computability/assignments

1 Question

Let A, B be two sets. Prove that the properties below are equivalent.

- $A = \emptyset \vee B = \emptyset$
- $A \times B = \emptyset$

1.1 Answer

$A \times B = \emptyset \iff \neg\exists(x,y).(x,y) \in A \times B \iff \forall(x,y).\neg(x \in A \wedge y \in B) \iff \forall(x,y).x \notin A \vee y \notin B \iff \forall x.x \notin A \vee \forall y.y \notin B \iff A = \emptyset \vee B = \emptyset$

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcup_{i=0}^{\infty} A_i = \bigcup\{A_i \mid i \in \mathbb{N}\}$ and $\bigcup_{i=0}^{k} A_i = \bigcup\{A_i \mid i \in \mathbb{N} \ \wedge \ i \leq k\} = A_0 \cup A_1 \cup \cdots \cup A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq \mathbb{N} \ (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or

- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$
2. p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$
3. p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$
4. p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

3.1 Answer

Answers + INTUITION of demonstration.

1. (*) is sufficient for property p_1 to hold.

Since $A_k$belongs to the union, it is simple to see that each one of its element will be in the union, therefore $\forall k \in \mathbb{N}.\ A_k \subseteq \bigcup_{i=0}^{k} A_i$. The equality relation requires us to show also that $\forall k \in \mathbb{N}.\ A_k \supseteq \bigcup_{i=0}^{k} A_i$but that follows by definition of Union and by the transitivity of the $\subseteq$ relation $(\forall i \in [0, k-1].\neg\exists x.x \in A_i \wedge x \notin A_k)$.

2. (*) is not sufficient to decide anything about property p_2.

It may be false: Let's give an example, defining $A_i = \{x | (\exists j.j \in [2, i+2] \wedge \exists k.k \in \mathbb{N}).j\ is\ prime \wedge x = j \times k\}$, we obtain a chain of infinite sets. It is easy to see that $A_i \subseteq A_{i+1}$ thus we satisfy property (*), and yet we will have INFINITELY OFTEN that $A_i \neq A_{i+1}$, since primes are infinite and uniformly distributed over $\mathbb{N}$. (RZ: OK, the "infinitely often" part was more general than was strictly necessary for a counterexample)

It may be true: pick $A_0 = \mathbb{N}$, then $A_0 = A_1 = A_2 = ...$, since all sets are subsets of $\mathbb{N}$, but we can not add any new element.

3. (*) is not sufficient to decide anything about property p_3.

It may be false: Let's define $A_i = \{x | x\, is\, even \wedge x \leq 2i\}$. Both the properties $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$ and (*) hold, but $\bigcup_{i=0}^{\infty} A_i$ is just the set of the even numbers, not equal to $\mathbb{N}$.

It may be true: the obvious example comes by construction of $\mathbb{N}$ itself and the $\leq$relation.

4. (*) is not sufficient to decide anything about property p_4.

It may be false: see the "may be false" case in the previous answer.

It may be true: see the "may be false" case in the next answer.

5. (*) is not sufficient to decide anything about property p_5.

It may be false: We may easily construct a chain of finite sets such that there exists an index k for which $A_k$is finite and is a FIXED POINT , which means that $A_i = A_{i+1} \forall i.i \in \mathbb{N} \wedge i \geq k$.

It may be true: We may easily construct a chain of finite sets for which we happen to have INFINITELY OFTEN an index k such that $|A_k| > |A_{k-1}|$ ($A_k \supset A_{k-1}$), which would be sufficient to show that the property holds. [i.e. A_i is the set of prime numbers less than or equal to index i]

6. (*) is sufficient for property p_6 to hold.

Obviously true, Q.E.D.

Viger