Computability Assignment
Year 2012/13 - Number 2

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1 Question

Let A, B be two sets. Prove that the properties below are equivalent.

- $A = \emptyset \vee B = \emptyset$
- $A \times B = \emptyset$

1.1 Answer

Start relating the two properties.

$A = \emptyset \vee B = \emptyset \iff A \times B = \emptyset$

Then I expand the Cartesian Product.

$A = \emptyset \vee B = \emptyset \iff \{< a, b > | \, a \in A \wedge b \in B\} = \emptyset$

Now I rewrite the first part using the cardinality.

$|A| = 0 \vee |B| = 0 \iff \{< a, b > | \, a \in A \wedge b \in B\} = \emptyset$

Now the Cartesian Product $(A \times B)$ will be empty since I cannot have any pair because I cannot find an $a \in A$ (or $b \in B$) since A (or B) is empty.

RZ: the last line proves only $\implies$, right?

2 Preliminaries

Given an infinite sequence of sets $(A_i)_{i\in\mathbb{N}}$, we define $\bigcup_{i=0}^{\infty} A_i = \bigcup\{A_i \mid i \in \mathbb{N}\}$ and $\bigcup_{i=0}^{k} A_i = \bigcup\{A_i \mid i \in \mathbb{N} \wedge i \leq k\} = A_0 \cup A_1 \cup \cdots \cup A_k$.

3 Question

Assume $(A_i)_{i\in\mathbb{N}}$ to be an infinite sequence of sets of natural numbers, satisfying

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \dots \subseteq \mathbb{N}\ (*)$$

For each property p_i shown below, state whether

- the hypothesis $(*)$ is sufficient to conclude that p_i holds; or
- the hypothesis $(*)$ is sufficient to conclude that p_i does not hold; or
- the hypothesis $(*)$ is not sufficient to conclude anything about the truth of p_i.

Justify your answers (briefly).

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$
2. p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$
3. p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$
4. p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite
5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite
6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite

3.1 Answer

1. p_1: $\forall k \in \mathbb{N}.\ A_k = \bigcup_{i=0}^{k} A_i$
   The hypothesis $(*)$ is sufficient to conlcude that p_1 holds.
   This is infer by the definition of the hypothesis itself $(*)$, since A_k already contains all the sets A_i, $i \leq k$, then the union does not add any new elements to A_k.

2. p_2: for all i, if A_i is infinite, then $A_i = A_{i+1}$
   The hypothesis $(*)$ is not sufficient to conlcude anything about the thruth of p_2.
   There exists only one case (RZ: actually more cases are possible) in which the property is true, which is when all the sets $A_i = \mathbb{N}$. I can also make the property false, if we consider $A_i = \mathbb{N} \setminus \{0\}$ and $A_{i+1} = \mathbb{N}$, the two sets are both infinte but they are not equal. Since I don't know what is contained in the sets I cannot conlcude anything about the property p_2.

3. p_3: if $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, then $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$
   The hypothesis $(*)$ is not sufficient to conlcude anything about the thruth of p_3.
   There exists only one case in which I get the $\bigcup_{i=0}^{\infty} A_i = \mathbb{N}$ with $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$, and is the case when all the set $A_i = \{a \in \mathbb{N} \,|\, a \leq i\}$ (RZ: more cases are possible). I can also make the property false, if we consider $A_i = \{2 * a \in \mathbb{N} \,|\, a \leq i\}$ we mantain the condition that $\forall i \in \mathbb{N}.\ A_i \neq A_{i+1}$,

but the $\bigcup_{i=0}^{\infty} A_i \neq \mathbb{N}$ (because the union of the A_i contains only the even number). Since I don't know what is contained in the sets I cannot conlcude anything about the property p_3.

4. p_4: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is finite
   The hypothesis $(*)$ is sufficient to conlcude that p_4 holds. (RZ: no)
   I know that $\forall i \in \mathbb{N}.\ |A_i| < \infty$, so, by contraddiction, if the $|\bigcup_{i=0}^{\infty} A_i| = \infty$ this would means (no) that at least one $|A_i| = \infty$ which contradicts the first condition of the property.

5. p_5: if $\forall i \in \mathbb{N}.\ A_i$ is finite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite
   The hypothesis $(*)$ is sufficient to conlcude that p_5 does not holds. (no)
   This can be proved using the proof gives in the point 4.

6. p_6: if $\forall i \in \mathbb{N}.\ A_i$ is infinite, then $\bigcup_{i=0}^{\infty} A_i$ is infinite
   The hypothesis $(*)$ is sufficient to conlcude that p_6 holds.
   I know that $\forall i \in \mathbb{N}.\ |A_i| = \infty$, so I'll surely got that the $|\bigcup_{i=0}^{\infty} A_i| = \infty$ because I need just one set $|A_i| = \infty$ to make the property true, as shown in the point 4.