Type Theory – 2017-06-20

Exercise 1. *Prove that, in a cartesian closed category, the following naturality law holds, for all objects A, B, C, D and all morphisms $f : A \to B$ and $g : B \times C \to D$.*

$$\Lambda((f \times \mathsf{id}_C); g) = f; \Lambda g$$

Exercise 2. *Replace the placeholders below so to make the following typing judgments correct, in the Calculus of Constructions.*

$$\begin{array}{ll}
1) & \alpha : *, P : \alpha \to *, Q : \alpha \to * \vdash \boxed{?} : \\
& (\prod_{a:\alpha}(Pa \to Qa)) \to \\
& (\prod_{\beta:*}\prod_{a:\alpha}((Pa \to \beta) \to \beta)) \to \\
& (\prod_{\beta:*}\prod_{a:\alpha}((Qa \to \beta) \to \beta)) \\
2) & \alpha : *, f : * \to *, x : \boxed{?} \vdash \\
& \lambda y : \boxed{?}.\ x\alpha(x(f\alpha)y) : \\
& f(f\alpha) \to \alpha
\end{array}$$

Exercise 3. *Prove the following properties on System F types, assuming parametricity. You can exploit known results, as long as you mention what you use. (Below, we let $2 = 1 + 1$, $T^2 = 2 \to T$ and associate $+, \times$ to the left.)*

$$\begin{array}{ll}
1) & (\alpha + \beta)^2 \simeq \alpha^2 + 2 \times \alpha \times \beta + \beta^2 \\
2) & \forall \alpha.\alpha \to (\alpha \times \alpha) \simeq 1
\end{array}$$

Then, explicitly provide λ terms for such isomorphisms and their inverses. (You are not required to prove these terms are isomorphisms.)

Exercise 4. *Let τ, σ be two System F type variables. Define a System F encoding T of the recursive type*

$$\mu X.1 + (\tau \to (X \times \sigma)) \times X$$

Then, find a type $U \simeq T$ such that U is of the form $\forall \alpha_1, \ldots, \alpha_n.U'$ with $n \geq 0$ and U' involving only type variables $\tau, \sigma, \alpha_1, \ldots$ and $\to$ types.

Exercise 5. *Consider the standard interpretation of the simply-typed λ calculus in a cartesian closed category $\mathcal{C}$. Define the morphism in $\mathcal{C}$ associated to the following typing judgment, where τ, σ are basic types.*

$$x : \tau \to \sigma \times \tau \vdash \lambda y : \tau.\pi_1(x(\pi_2(xy))) : \tau \to \sigma$$

Exercise 6. *Let $\mathcal{C}$ be a category. Define the category $\mathcal{C}^\to$ as the one having as objects the morphisms of $\mathcal{C}$, and as morphisms the "commuting squares" in $\mathcal{C}$. More precisely:*

$$\begin{array}{l}
|\mathcal{C}^\to| = \{(A, B, f) \mid A, B \in |\mathcal{C}| \land f : A \to_{\mathcal{C}} B\} \\
\mathcal{C}^\to((A, B, f), (C, D, g)) = \{(m : A \to_{\mathcal{C}} C, n : B \to_{\mathcal{C}} D) \mid f; n = m; g\}
\end{array}$$

1. *Precisely define the composition of morphisms in the "obvious" way.*
2. *Verify that $\mathcal{C}^\to$ is indeed a category.*
3. *Let $\mathcal{D}$ be the two-points preorder $\bot \sqsubseteq \top$, seen as a category. Prove that the category $[\mathcal{D}, \mathcal{C}]$ (which has functors as objects and natural transformations as morphisms) is isomorphic to $\mathcal{C}^\to$.*