Type Theory – A few exercises

Exercise 1. *Fill the holes in the following judgments so that they hold in the Calculus of Constructions.*

$$\begin{array}{l}
\alpha : * \vdash \boxed{?} : \alpha \to \alpha \\
\alpha : * \vdash \alpha \to \boxed{?} : \square \\
\alpha : *, a : \alpha, \beta : *, b : \beta, x : \boxed{?} \vdash \langle x \boxed{?} a, x \boxed{?} b \rangle : \alpha \times \beta \\
\vdash \boxed{?} : \prod_{\alpha:*} \prod_{\beta:*} (\alpha \to \beta) \to (\alpha \times \alpha) \to (\beta \times \beta) \\
\alpha : *, f : \alpha \to \alpha, P : \alpha \to * \vdash \boxed{?} : \\
\qquad (\prod_{a:\alpha} Pa \to P(fa)) \to \prod_{x:\alpha} Px \to P(f(fx)) \\
\vdash \boxed{?} : \prod_{\alpha:*} (((\alpha \to \bot) \to \bot) \to \bot) \to \alpha \to \bot \\
\alpha : *, x : \alpha, h : \prod_{P:\alpha \to *} (Px \to \bot) \to \bot \vdash \boxed{?} : \bot
\end{array}$$

where $\bot = \prod_{\alpha:*} \alpha$

Exercise 2. *Prove that, in a cartesian closed category,* 1×1 *is final.*

Exercise 3. *Prove that, in a category with an initial object* 0*, if* $X \simeq 0$*, then* X *is initial.*

Exercise 4. *Refute the following, using a counterexample. In every category with products,* $1 + 1$ *is final.*

Exercise 5. *Let* $\mathcal{C}$ *be a category with binary products and coproducts, and an object* A*. Consider the following functors* $\mathcal{C} \to \mathcal{C}$*:*

$$FX = X \times X \qquad GX = X + X \qquad HX = X \times A \qquad LX = X + A$$

Define how they act on morphisms, using $\langle -, - \rangle$ *and* $[-, -]$ *suitably.*

Exercise 6. *In a bicartesian closed category, prove that, for any objects* A, B, C*:*

$$A \times (B + C) \simeq (A \times B) + (A \times C)$$

Hint: for direction $\to$*, start by defining a morphism*

$$B + C \to ((A \times B) + (A \times C))^A$$

Exercise 7. *Consider a polymorphic* λ *calculus with a list (or finite sequence) type* α^**. Let* f *be a term satisfying*

$$\vdash f : \forall \alpha. (\alpha \times \alpha^*) \to \alpha$$

and consider the interpretation of f *in a parametric model.*

Denote with $\langle\rangle$ *the empty list in* α^**. Prove that, in the model,* $f_\alpha x \langle\rangle = x$*, for any* α *and any value* x *in the interpretation of* α*.*

Exercise 8. *Prove that in any category with binary products, define a natural isomorphism* $f : A \times B \simeq B \times A$*. Then, verify that* f *is indeed an isomorphism (for any* A, B*), and that it is indeed natural.*

Exercise 9. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor. Prove that* F *maps isomorphic objects (in* $\mathcal{C}$*) into isomorphic objects (in* $\mathcal{D}$*).*

Assuming F *is fully faithful (recall the definition from the course notes), prove that the opposite implication holds as well: if* $FA \simeq FB$*, then* $A \simeq B$*.*

Exercise 10. *In a cartesian closed category, let* $f, g : A \times B \to C$*. Assuming* $\Lambda f = \Lambda g$*, prove* $f = g$*.*

Exercise 11. *In a category with binary products, let* $f_1, f_2 : A \to B, g_1, g_2 : A \to C$*. Assuming* $\langle f_1, g_1 \rangle = \langle f_1, g_1 \rangle$*, prove* $f_1 = f_2$ *and* $g_1 = g_2$*.*

Exercise 12. *In a cartesian closed category, prove that for any object* A*,*

$$\langle \Lambda(\langle \pi_2^{A \times A^A}, \pi_1^{A \times A^A} \rangle ; \mathsf{apply}^{A^A}), \Lambda \pi_2^{A \times A} \rangle ; \mathsf{apply}^{A^{(A^A)}} = id_A$$

Exercise 13. *Consider the standard interpretation of the simply-typed* λ *calculus in a cartesian closed category* $\mathcal{C}$*. Define the morphism in* $\mathcal{C}$ *associated to the following typing judgment, where* τ *is a basic type.*

$$f : \tau \to \tau, x : \tau \vdash f(fx) : \tau$$

Exercise 14. *Let* $\mathcal{C}$ *be a category with an object* A*. Formally define the* slice category $\mathcal{C}/A$*, informally described as follows, and prove it is indeed a category. The objects of* $\mathcal{C}/A$ *are the morphisms* $X \to A$ *in* $\mathcal{C}$*, for some object* X*. The morphisms between* $f : X \to A$ *and* $g : Y \to A$ *are those morphisms* $h : X \to Y$ *making the obvious triangle commute.*

Exercise 15. *Prove that, in a slice category* $\mathcal{C}/A$ *(see above for the definition), the following holds.*

- $\mathcal{C}/A$ *always has a final object.*
- *If* $\mathcal{C}$ *has an initial object, so does* $\mathcal{C}/A$*.*
- *If* A *is final in* $\mathcal{C}$*, then* $\mathcal{C}/A \simeq \mathcal{C}$*.*