Computability Final Test — 2014-06-12

Notes.

- Answer both theory questions, and choose and solve <u>two</u> exercises, only. Solving more exercises results in the <u>failure</u> of the test.
- To pass the exam you need to provide a reasonable contribution in *both* Theory and Exercises parts.
- Exercises with higher number award more points. To achieve a score ≥ 28 you have to solve an exercise marked with $\star$ below.
- Significantly wrong answers will result in negative scores.
- <u>Always provide a justification for your answers.</u>

Reminder: when equating the results of partial functions (as in $\phi_i(3) = \phi_i(5)$), we mean that <u>either</u> 1) both sides of the equation are defined, and evaluate to the same natural number, <u>or</u> 2) both sides are undefined.

Theory

Question 1. *Prove Cantor's theorem stating that there is no bijection between A and $\mathcal{P}(A)$ for any set A.*

Question 2. *State and prove the second recursion theorem.*

Exercises

Exercise 3. *Prove whether*

$$A = \{n + 1 \mid \forall x.\ \phi_n(2 \cdot x + 1) = 5\} \in \mathcal{RE}$$

Solution (sketch). Let $B = \{n \mid \forall x.\ \phi_n(2 \cdot x + 1) = 5\}$ is $\mathcal{RE}$. We have $B \leq_m A$ with reduction $h(n) = n + 1$ (easy to check). We now prove $B \notin \mathcal{RE}$, which enables to conclude that $A \notin \mathcal{RE}$ exploiting the previous reduction.

To prove $B \notin \mathcal{RE}$, we apply Rice-Shapiro ($\Rightarrow$). B is semantically closed (easy). Let $\mathcal{F}$ be the associated set of functions. Take $f(n) = 5$. Clearly f is recursive and belongs to $\mathcal{F}$. Let g be any arbitrary finite-domain restriction of f. We have $g \notin \mathcal{F}$, since otherwise g would be defined on all the points of the form $2 \cdot x + 1$, i.e. on all odd naturals, hence on infinitely many points – contradicting the assumption that $\mathsf{dom}(g)$ is finite. □

Exercise 4. *Prove whether*

$$B = \{n \mid \forall x.\ \phi_n(x + 1) = x \vee x \geq 3\} \leq_m \mathsf{K}$$

Solution (sketch). The statement is true. Since K is $\mathcal{RE}$-complete, it suffices to prove that $B \in \mathcal{RE}$. We have

$$B = \{n \mid \forall x < 3.\ \phi_n(x+1) = x\}$$

That is,

$$B = \{n \mid \phi_n(1) = 0 \land \phi_n(2) = 1 \land \phi_n(3) = 2\}$$

The last property is a conjunction of three parts. If we prove these parts to be $\mathcal{RE}$ predicates of n, we can conclude that the set B is $\mathcal{RE}$. Indeed, proving that the first part is $\mathcal{RE}$ can be done by writing a semiverifier as follows:

$S_0(n)$:
run $\phi_n(1)$
take its result z
if $z \neq 0$, loop forever
return 1

It is easy to check that the above is indeed a semiverifer. Writing the semiverifiers for the other two predicates is done in a similar way (only numeric constants change). □

Exercise 5. *Prove that we do* **not** *have*

$$C = \{n \mid \exists x \in \mathbb{N}.\ \phi_n(x^2) \text{ is defined}\} \leq_m \bar{\mathsf{K}}$$

Solution (sketch). If the above were true, we would also have, by complementing both sides:

$$\bar{C} = \{n \mid \forall x \in \mathbb{N}.\ \phi_n(x^2) = undefined\} \leq_m \mathsf{K}$$

The above implies that $\bar{C} \in \mathcal{RE}$. However, we can prove $\bar{C} \notin \mathcal{RE}$ by Rice-Shapiro ($\Leftarrow$). Indeed, $\bar{C}$ is semantically closed (easy to check). Let then $\mathcal{F}$ be its associated set of functions. The always undefined function $g(n) = undefined$ clearly belongs to $\mathcal{F}$, and is finite. The identity function $f(n) = n$ is a recursive extension of g and we also have $f \notin \mathcal{F}$ since e.g. $f(2^2) = 4 \neq undefined$. Hence, $\bar{C} \notin \mathcal{RE}$. □

Exercise 6. ⋆ *Consider the following set of total functions*

$$\mathcal{H} = \left\{ h \in (\mathbb{N} \to \mathbb{N}) \;\middle|\; \forall n, x \in \mathbb{N}. \begin{pmatrix} \phi_n \in \mathcal{PR} \implies \phi_{h(n)}(x) = x^2 + 1 \\ \land \\ \phi_n \notin \mathcal{PR} \implies \phi_{h(n)}(x) = 2 \cdot x + 10 \end{pmatrix} \right\}$$

Prove whether $\mathcal{H} = \emptyset$ and whether $\mathcal{H} \cap \mathcal{R} = \emptyset$.

Solution (sketch). Intentionally omitted. □