Computability Final Test — 2014-01-13

Notes.

- Answer both theory questions, and choose and solve two exercises, only. Solving more exercises results in the failure of the test.
- To pass the exam you need to provide a reasonable contribution in *both* Theory and Exercises parts.
- Exercises with higher number award more points. To achieve a score ≥ 28 you have to solve an exercise marked with $\star$ below.
- Significantly wrong answers will result in negative scores.
- Always provide a justification for your answers.

Reminder: when equating the results of partial functions (as in $\phi_i(3) = \phi_i(5)$), we mean that either 1) both sides of the equation are defined, and evaluate to the same natural number, or 2) both sides are undefined.

Theory

Question 1. *Use a diagonalization argument to prove that a partial function (of your choice) is not recursive.*

Answer. See for instance Th. 37 (as suggested at the bottom of page 115), or alternatively Lemma 140 (since λ-definability is equivalent to recursivity).

Question 2. *Prove that, if* $\exists f \in \mathcal{R}.\ A = \mathsf{ran}(f)$, *then* $A \in \mathcal{RE}$.

Answer. See Lemma 246.

Exercises

Exercise 3. *Prove whether* $A \in \mathcal{RE}$, *where:*

$$A = \{n \mid \phi_n(0) = 9 \wedge \forall x \in \mathbb{N}.\ \phi_n(x) = \phi_n(x + 10)\}$$

Then, briefly discuss if your reasoning still holds if we remove the "$\phi_n(0) = 9$*" requirement from the definition above.*

Solution (sketch). $A \notin \mathcal{RE}$ by Rice-Shapiro ($\Rightarrow$). Indeed, A is semantically closed (easy to check). Let $\mathcal{F}$ be the associated set of functions. If we consider $f(n) = 9$, we clearly have $f \in \mathcal{F}$. However, any finite restriction g of f can not belong to $\mathcal{F}$ because in that case it would be defined ($= 9$) on all the multiples of 10, which are infinitely many.

If we remove the "$\phi_n(0) = 9$" requirement, instead, $g(n) = undefined$ would be a finite restriction of f belonging to $\mathcal{F}$. Hence, the previous reasoning would not hold (and we should instead use Rice-Shapiro in the other direction). $\square$

Exercise 4. *Let* $B = \{n|\phi_n(2) = undefined \wedge \phi_n(3) = 3\}$*. Prove that, for all* $A \in \mathcal{RE}$*,* $A \leq_m B$*.*

Solution (sketch). Since K is $\mathcal{RE}$-complete, it suffices to prove $\mathsf{K} \leq_m B$. We take

$$h(n) = \#\left(\lambda x. \begin{cases} 3 & \text{if } x = 3 \wedge n \in \mathsf{K} \\ undefined & \text{otherwise} \end{cases}\right)$$

The body of the λx is partial recursive by the if-then-else lemma with $\mathcal{RE}$ guard. Hence, by the s-m-n theorem h is total recursive. We now check it is indeed a reduction.

- If $n \in \mathsf{K}$, then

  $$\phi_{h(n)}(x) = \begin{cases} 3 & \text{if } x = 3 \wedge true \\ undefined & \text{otherwise} \end{cases} = \begin{cases} 3 & \text{if } x = 3 \\ undefined & \text{otherwise} \end{cases}$$

  In particular, $\phi_{h(n)}(3) = 3$ and $\phi_{h(n)}(2) = undefined$, hence $h(n) \in B$.

- If $n \notin \mathsf{K}$, then

  $$\phi_{h(n)}(x) = \begin{cases} 3 & \text{if } x = 3 \wedge false \\ undefined & \text{otherwise} \end{cases} = undefined$$

  In particular, $\phi_{h(n)}(3) = undefined \neq 3$, hence $h(n) \notin B$.

□

Exercise 5. *State whether* $f \in \mathcal{R}$ *where*

$$f(n) = \begin{cases} n^2 & \textit{if } \phi_n(2) \textit{ is defined} \\ 4 \cdot n + 5 & \textit{otherwise} \end{cases}$$

Solution (sketch). We prove $f \notin \mathcal{R}$ by contradiction. Assuming $f \in \mathcal{R}$, since f is total it is easy to construct a verifier for $A = \{n|f(n) = n^2\}$. Hence, $A \in \mathcal{R}$. According to the definition of f we have that $A = \{n \mid \phi_n(2) \text{ defined } \vee n = 5\}$ (note that $n^2 = 4 \cdot n + 5$ iff $n = 5$ when $n \in \mathbb{N}$). Let $B = \{5\}$ if $\phi_5(2)$ is undefined, and $B = \emptyset$ otherwise. In any case, B is finite, hence recursive. We then consider the set $C = A \setminus B = \{n|\phi_n(2) \text{ defined}\}$ which is recursive bcause it is a difference of $\mathcal{R}$ sets. But then, by Rice we can prove $C \notin \mathcal{R}$ – contradiction. □

Exercise 6. ⋆ *Prove that any infinite set* $A \in \mathcal{RE}$ *includes an infinite set* $B \in \mathcal{R}$*.*
Some hints are provided below:

- *[1% score] Rewrite "*$A \in \mathcal{RE}$*" in a different but equivalent way.*
- *[1% score] Prove that, if* $f \in \mathcal{R}$ *is total, then* $g(n) = max\{f(x) \mid 0 \leq x \leq n\}$ *is recursive and monotonic (*$n \leq m \implies g(n) \leq g(m)$*).*
- *[98% score] Define* B *exploiting the above and conclude.*

Solution (sketch). Intentionally omitted. □