Computability Final Test — 2012-07-10

Notes.

- Write your name and matriculation number on each of your sheets.
- Solve no more than four (4) exercises. This will be strictly enforced: including more than 4 answers will result in the immediate failure of the test.
- Significantly wrong answers will result in negative scores.
- Always provide a justification for your answers.
- To achieve a score ≥ 27 you have to solve the exercise marked with $\star$ below.

Reminder: when equating results of partial functions (as in $\phi_i(3) = \phi_i(5)$), we mean that either 1) both sides of the equation are defined, and evaluate to the same natural number, or 2) both sides are undefined.

Exercise 1. *Prove whether* $A = \{i \mid \phi_i(3) > \phi_{i+1}(5)\} \in \mathcal{RE}$.

Solution (sketch). $A \in \mathcal{RE}$ because the following is a semi-verifier:

$$S_A = \lambda i.\ \mathbf{Lt}\ (\mathbf{Eval1}\ (\mathbf{Succ}\ i)\ \ulcorner 5 \urcorner)\ (\mathbf{Eval1}\ i\ \ulcorner 3 \urcorner)\ \mathbf{I}\ \Omega$$

Checking that the above halts exactly on $i \in A$ is easy. Note that when $\phi_i(3)$ or $\phi_{i+1}(5)$ are undefined the program loops inside the "**Eval1**" part (and never reaches $\mathbf{I}$ or Ω). However, in this cases $i \notin A$, since $undefined$ is not less or greater than any value (including $undefined$ itself). Hence the fact that S_A does not halt in such cases is indeed the wanted behaviour. □

Exercise 2. *Prove whether* $\bar{\mathsf{K}} \leq_m B = \{i \mid \exists n.\ \phi_i(n) = n + 1\}$.

Solution (sketch). We have $\bar{\mathsf{K}} \not\leq_m B$ because $\bar{\mathsf{K}} \notin \mathcal{RE}$ and $B \in \mathcal{RE}$. The latter can be proved as follows. The property $p(i, n) =$ "$\phi_i(n) = n + 1$" is $\mathcal{RE}$ since it can be semi-verified by

$$S_p = \lambda i\, n.\ \mathbf{Eq}\ (\mathbf{Eval1}\ i\ n)\ (\mathbf{Succ}\ n)\ \mathbf{I}\ \Omega$$

(Checking that S_p halts exactly when p holds is easy.) Hence $\exists n.\ p(i, n)$ is $\mathcal{RE}$ since it is an existential quantification of an $\mathcal{RE}$ property. This proves $B \in \mathcal{RE}$. □

Exercise 3. *Comment on this statement by Mr. Rouge Hareng: is it correct?*

Let h be a recursive partial function. If $A \subseteq B$, then $f \subseteq g$ where

$$f(n) = \begin{cases} h(n) + 1 & \text{if } n \in A \\ undefined & \text{otherwise} \end{cases} \qquad g(n) = \begin{cases} h(n) + 1 & \text{if } n \in B \\ h(n) & \text{otherwise} \end{cases}$$

Solution (sketch). Yes, it is correct. We just have to prove that

$$\forall n \in \mathsf{dom}(f).\ f(n) = g(n)$$

Note that by definition f is undefined outside A, hence $\mathsf{dom}(f) \subseteq A$. The above goal is then implied by

$$\forall n \in A.\ f(n) = g(n)$$

When $n \in A$, we have $f(n) = h(n)+1$. Also, since $n \in A \subseteq B$, $g(n) = h(n)+1$. Therefore $f(n) = g(n)$. □

Exercise 4. *Prove whether* $C = \{i \mid \forall n.\ (n\ \textit{prime} \implies \phi_i(n) = 3)\} \in \mathcal{RE}$.

Solution (sketch). $C \notin \mathcal{RE}$ by Rice-Shapiro ($\Rightarrow$). C is semantically closed (easy to check). Take $f(x) = 3$ for all x. Clearly, $f \in \mathcal{F}_C$. Consider now any finite restriction g of f. Since the domain of g is finite, and primes are infinite, there is a prime $p \notin \mathsf{dom}(g)$. Hence, we have $g(p) = \textit{undefined} \neq 3$ which proves $g \notin \mathcal{F}_C$. We conclude $C \notin \mathcal{RE}$. □

Exercise 5. *Prove whether* $D = \{i \mid \mathsf{dom}(\phi_{\mathsf{proj1}(i)}) \subseteq \mathsf{dom}(\phi_{\mathsf{proj2}(i)})\} \in \mathcal{RE}$.

Solution (sketch). $D \notin \mathcal{RE}$ since $Tot \leq_m D$, where $Tot = \{i \mid \mathsf{dom}(\phi_i) = \mathbb{N}\} \notin \mathcal{RE}$ by Rice-Shapiro ($\Rightarrow$). A possible reduction is

$$h(n) = \mathsf{pair}(a, n) \quad \text{where } a \text{ is such that } \phi_a = \mathsf{id}$$

Verifying the above claims is left as an exercise. □

Exercise 6. *Prove whether* $f \in \mathcal{R}$, *where*

$$f(n) = \begin{cases} 2 \cdot n + 1 & \textit{if } \phi_n(0) = 0 \\ 44 & \textit{otherwise} \end{cases}$$

Solution (sketch). We prove $f \notin \mathcal{R}$ as follows:
Let $A = \{n \mid \phi_n(0) = 0\}$. We have that

$$n \in A \iff f(n) \text{ odd}$$

Indeed, if $n \in A$, then $f(n) = 2 \cdot n + 1$ which is odd; otherwise, if $n \notin A$, then $f(n) = 44$.

By contradiction, assume $f \in \mathcal{R}$. Then $A \in \mathcal{R}$ since by the above equivalence, we can verify $n \in A$ by computing $f(n)$ and checking for oddness. However, $A \notin \mathcal{R}$ by Rice (easy to check). □

Exercise 7. *Prove whether* $\bar{\mathsf{K}} \leq_m E = \{i \mid \exists n.\ \forall m > n.\ \phi_i(m) = m + 1\}$.

Solution (sketch). $\bar{\mathsf{K}} \leq_m E$. Indeed, take

$$h(n) = \#\left(\lambda x. \begin{cases} x + 1 & \text{if } \phi_n(n) \text{ does not halt in} \leq x \text{ steps} \\ 0 & \text{otherwise} \end{cases}\right)$$

The above is well-defined because the guard is recursive and the branches are also such. Then:

- If $n \in \bar{\mathsf{K}}$, then $\phi_n(n)$ does not halt, hence $\phi_{h(n)}(x) = x + 1$ for all x, and so $h(n) \in E$.
- If $n \notin \bar{\mathsf{K}}$, then $\phi_n(n)$ halts, say in k steps. Hence, $\phi_{h(n)}(x) = 0$ for all $x > k$. Assume by contradiction $h(n) \in E$: then we have $\phi_{h(n)}(x) = x+1$ for all $x > m$, for some m. Take any x larger than m and k. We get $\phi_{h(n)}(x) = 0 = x + 1$ which is a contradiction. Hence $h(n) \notin E$.

□

Exercise 8. *We write* $\min A$ *for the minimum element of a set* A.
State whether a partial *recursive function* f *exists such that,*

$$\forall i \in \mathbb{N}.\ \forall A \subseteq \mathbb{N}.\ (A \neq \emptyset \land \phi_i = \chi_A \implies f(i) = \min A)$$

State whether a total *recursive function* g *exists such that,*

$$\forall i \in \mathbb{N}.\ \forall A \subseteq \mathbb{N}.\ (A \neq \emptyset \land \phi_i = \chi_A \implies g(i) = \min A)$$

Solution (sketch). Intentionally omitted. □

Exercise 9. ⋆ *(This exercise counts as two exercises. If you choose it, solve only 3 exercises instead of 4.) Prove that*

$$\exists A \subseteq \mathbb{N}.\ A \text{ infinite } \land \left(\nexists B \in \mathcal{RE}.\ B \text{ infinite } \land B \subseteq A\right)$$

(Hint: construct A *by diagonalization)*

Solution (sketch). Intentionally omitted. □