Computability Final Test — 2012-01-19

Notes.

- Write your name and matriculation number on each of your sheets.
- Solve no more than four (4) exercises. This is strictly enforced: including more than 4 answers results in the immediate failure of the test.
- Significantly wrong answers result in negative scores.
- Always provide a justification for your answers.
- To achieve higher scores (≥ 27) you have to solve some exercises marked with $\star$ below.

Reminder: when equating results of partial functions (as in $\phi_i(3) = \phi_i(5)$), we mean that either 1) both sides of the equation are defined, and evaluate to the same natural number, or 2) both sides are undefined.

Exercise 1. *Comment on this statement by Mr. Rouge Hareng: is it correct? If so, help Mr. Hareng in writing a more detailed proof; otherwise, provide a counterexample.*

> *If A is a subset of B, we have that $A \leq_m B$ trivially holds: it suffices to take as a reduction any total recursive function whose range is B.*

Solution (sketch). $\neg(\mathsf{K} \leq_m \mathbb{N})$ is a counterexample. If $\mathsf{ran} f = B$, it is then impossible to prove "if $n \notin A$ then $f(n) \notin B$". □

Exercise 2. *State whether* $A = \{i \mid \forall x \in \mathbb{N}.\ (\phi_i(x) = 5 \iff x \textit{ is even})\}$ *is* $\mathcal{RE}$.

Solution (sketch). A is semantically closed (easy to check). By Rice-Shapiro ($\Rightarrow$): let $f(n) = 5$ for even n, and $f(n) = 6$ otherwise. This f is recursive and $f \in \mathcal{F}_A$. However, no finite restriction g of f belongs to $\mathcal{F}_A$ since in that case it would be defined on all even numbers, hence on infinitely many points. Hence $A \notin \mathcal{RE}$. □

Exercise 3. *State whether* $B = \{i \mid \exists x.\ x \in \mathsf{K} \cap \mathsf{ran}(\phi_i)\}$ *is* $\mathcal{RE}$.

Solution (sketch). $B \in \mathcal{RE}$. Indeed, the predicate defining B can be equivalently rewritten as $\exists x, w, k_1, k_2 . q(x, w, k_1, k_2)$ where $q(x, w, k_1, k_2)$ is defined as

$$\phi_x(x) \text{ halts in } k_1 \text{ steps, and } \phi_i(w) \text{ halts in } k_2 \text{ steps with result } x$$

which is recursive (one can use a step-by-step interpreter to verify it). Hence its existential quantification is $\mathcal{RE}$. $\square$

Exercise 4. *A set $C \subseteq \mathbb{N}$ is said to be upward closed iff*

$$\forall x \in C.\, \forall y \in \mathbb{N}.\, [y > x \implies y \in C]$$

Prove that any upward closed C is $\mathcal{RE}$.

Solution (sketch). If C is empty then it is $\mathcal{R}$, hence $\mathcal{RE}$. Otherwise, $C = \{x \mid k \leq x\}$ where $k = \min C$. So C can be verified by $V_C = \lambda x.\, \mathbf{Leq}\, \ulcorner k \urcorner\, x$. Hence C is $\mathcal{R}$, hence $\mathcal{RE}$. $\square$

Exercise 5. *Recall that a partial function $f \in \mathbb{N} \rightsquigarrow \mathbb{N}$ is* injective *iff*

$$\forall x, y \in \mathsf{dom}(f).\big[f(x) = f(y) \implies x = y\big]$$

Let $D = \{i \mid \phi_i \text{ injective}\}$. Answer these questions:

1) $D \in \mathcal{R}$?   2) $\overline{D} \in \mathcal{RE}$?   3) $D \in \mathcal{RE}$?

Solution (sketch).

- 1) $D \notin \mathcal{R}$ by Rice. (easy to check)
- 2) $\bar{D} \in \mathcal{RE}$ since

  $$\begin{aligned} \bar{D} &= \{i \mid \exists x, y.\, x, y \in \mathsf{dom}(\phi_i) \land \phi_i(x) = \phi_i(y) \land x \neq y\} \\ &= \{i \mid \exists x, y.\, \phi_i(x) = \phi_i(y) \neq \text{undefined} \land x \neq y\} \end{aligned}$$

  which is $\mathcal{RE}$ because it is an existential quantification of a $\mathcal{RE}$ property. Indeed, the latter is a conjunction of two $\mathcal{RE}$ properties:

  $$\begin{aligned} p(i, x, y) &: \quad \phi_i(x) = \phi_i(y) \neq \text{undefined} \\ q(i, x, y) &: \quad x \neq y \end{aligned}$$

  The second, q, is trivially recursive, hence $\mathcal{RE}$. Instead, p is $\mathcal{RE}$ since it can be semi-verified using a universal program as follows:

  $$\lambda i, x, y.\, \mathbf{Eq}\, (\mathbf{Eval1}\; i\; x)\, (\mathbf{Eval1}\; i\; y)\, \mathbf{I}\, \Omega$$

  Note that the above loops when $\phi_i(x)$ or $\phi_i(y)$ are not defined, but in that case $p(i, x, y)$ does not hold, so looping is the right thing to do.

- 3) $D \notin \mathcal{RE}$, otherwise we contradict 1) and 2).

$\square$

Exercise 6. *State whether $E = \{4^i \cdot 5^j \mid \forall x.\, \phi_i(x) = \phi_j(x+1)\}$ is $\mathcal{RE}$.*

Solution (sketch). $\bar{\mathsf{K}} \leq E$ with reduction

$$h(n) = 4^{f(n)} \cdot 5^a$$
$$\phi_a(n) = undefined \qquad \text{for all } n$$
$$f(n) = \#(\lambda x.\ \phi_n(n))$$

Function h is recursive and total (because ...). Also, $h(n) \in E$ is equivalent to $\forall x.\ \phi_{f(n)}(x) = \phi_a(x+1)$ (since $g(b,c) = 4^b \cdot 5^c$ is injective), which is equivalent to $\forall x.\ \phi_{f(n)}(x) = undefined$, which is equivalent to $\forall x.\ \phi_n(n) = undefined$, which is equivalent to $\phi_n(n) = undefined$, which is equivalent to $n \in \bar{\mathsf{K}}$. □

Exercise 7. *Construct a total function $f \in (\mathbb{N} \to \mathbb{N})$ such that*

1) $\forall n.\ f(n) > 2^n$ 2) $g(n) = \lfloor \frac{f(n)}{2} \rfloor$ is recursive 3) f is not recursive.

Solution (sketch). Let $f(n) = 2^{n+1} + \chi_{\mathsf{K}}(n)$. We have $f(n) \geq 2^{n+1} > 2^n$, hence 1) holds. Function $g(n) = \lfloor \frac{2^{n+1}+\chi_{\mathsf{K}}(n)}{2} \rfloor = \lfloor \frac{2^{n+1}}{2} \rfloor = 2^n$ is recursive, so 2) holds. Finally, $f \notin \mathcal{R}$ since otherwise $f(n) - 2^{n+1} = \chi_{\mathsf{K}}(n)$ would be recursive, contradicting $\mathsf{K} \notin \mathcal{R}$. Hence 3) holds. □

Exercise 8. *Let $A_0, A_1, \ldots$ be an infinite sequence of sets of natural numbers. Define a set B such that $\forall i.\ A_i \leq_m B$.*

Solution (sketch). Take $B = \{\mathsf{pair}(i,n) \mid i \in \mathbb{N} \wedge n \in A_i\}$. Then $h_i(n) = \mathsf{pair}(i,n)$ is a m-reduction from A_i to B. (because ...) □

Exercise 9. ⋆ *Prove whether there exists a bijection f between $\mathbb{N}$ and $\mathcal{RE}$. Then, prove whether there exists a bijection f between $\mathcal{RE}$ and $\mathcal{P}(\mathbb{N})$. You may wish to exploit the following property:*

(Schröder-Bernstein) There exists a bijection $f \in (A \leftrightarrow B)$ if and only if there exist injective *functions $g \in (A \to B)$ and $h \in (B \to A)$.*

Solution (sketch). Intentionally omitted. □

Exercise 10. ⋆ *Let $w \in (\Lambda \to \mathbb{N})$ be inductively defined as*

$$w(x_i) = 0 \qquad w(MN) = w(N) + w(M) \qquad w(\lambda x_i.\ M) = 1 + w(M)$$

Further, let $A = \{\#M \mid w(M) < 10^6\}$ and $B = \{i \mid \forall x.\ \phi_i(x) = x\}$. Assume that recursive function are enumerated using λ-terms, by letting ϕ_i to be the partial function computed by λ-term M with $\#M = i$. Formally, assume

$$\phi_{\#M}(x) = y \iff M \ulcorner x \urcorner =_{\beta\eta} \ulcorner y \urcorner$$

Then, answer the following questions:

1. *[2% score] Is A a recursive set?*
2. *[2% score] Is B a recursive set?*
3. *[96% score] Is $A \cap B$ a recursive set?*

Solution (sketch). Intentionally omitted. □