Computability Final Test — 2011-07-07

Notes.

- Write your name and matriculation number on each of your sheets.
- Solve no more than four (4) exercises. This will be strictly enforced: including more than 4 answers will result in the immediate failure of the test.
- Significantly wrong answers will result in negative scores.
- Always provide a justification for your answers.
- To achieve higher scores (≥ 27) you have to solve at least one exercise marked with $\star$ below.

Reminder: when equating results of partial functions (as in $\phi_i(3) = \phi_i(5)$), we mean that either 1) both sides of the equation are defined to be the same natural number, or 2) both sides are undefined.

Exercise 1. *State whether* $A = \{n \mid 2^n < 1000 \land \phi_n(n) = 2\} \in \mathcal{R}$.

Solution (sketch). The set A may contain only numbers < 10 because $2^{10+k} \geq 2^{10} = 1024 > 1000$. Hence it is a finite set. So, $A \in \mathcal{R}$. □

Exercise 2. *State whether* $B = \{\mathsf{proj2}(n) \mid \phi_n(6) = n\} \in \mathcal{RE}$.

Solution (sketch). $B \in \mathcal{RE}$. Indeed, let

$$f(n) = \begin{cases} \mathsf{proj2}(n) & \text{if } \phi_n(6) = n \\ undefined & o.w. \end{cases}$$

The above f is recursive because 1) the property "$\phi_n(6) = n$" is $\mathcal{RE}$ (because e.g. it has semi-verifier $\lambda n.$ **Eq** (**Eval** $n \ulcorner 6 \urcorner$) n), and 2) when the property holds $f(n)$ is $\mathsf{proj2}(n)$ which is recursive, and 3) when the property does not hold $f(n)$ is undefined. The range of f is clearly B, therefore $B \in \mathcal{RE}$. □

Exercise 3. *Comment on this proof excerpt by Mr. Rouge Hareng: is it correct? Why?*

> *... and therefore,* $A = \mathsf{dom}(f)$*. Since* f *is a recursive function, we obtain* $A = \mathsf{dom}(\phi_i)$ *for some* i*, hence* $A \in \mathcal{RE}$*, which contradicts the hypothesis* $A \in \mathcal{R}$*. This concludes our proof.*

Solution (sketch). The last step is wrong. $A \in \mathcal{RE}$ does not imply $A \notin \mathcal{R}$ (e.g. $\emptyset \in \mathcal{RE}$ but $\emptyset \in \mathcal{R}$). The rest of the proof is OK. □

Exercise 4. *Prove that, for any predicate* $P(-)$*:*

$$\Big(\exists n.\ P(n)\Big) \iff \Big(\exists x, y.\ P(\mathsf{pair}(x, y))\Big)$$

Then, prove that $A \in \mathcal{RE}$ *if and only if*

$$A = \{x \mid \exists y, z.\ \mathsf{pair}(x, \mathsf{pair}(y, z)) \in B\} \text{ for some } B \in \mathcal{R}$$

Solution (sketch). Part 1. ($\Rightarrow$) If for some n we have $P(n)$, we can take $x = \mathsf{proj1}(n), y = \mathsf{proj2}(n)$ and have $\mathsf{pair}(x, y) = n$, so $P(\mathsf{pair}(x, y))$. (Note: an alternative solution would exploit the fact that pair is a surjective function.)

($\Leftarrow$) If for some x, y we have $P(\mathsf{pair}(x, y))$, we can take $n = \mathsf{pair}(x, y)$ and have $P(n)$.

Part 2. We know that $A \in \mathcal{RE}$ if and only if there is a $B \in \mathcal{R}$ such that

$$A = \{m \mid \exists n.\ \mathsf{pair}(m, n) \in B\}$$

By Part 1, taking $P(n) =$ "$\mathsf{pair}(m, n) \in B$" the above is equivalent to

$$A = \{m \mid \exists x, y.\ \mathsf{pair}(m, \mathsf{pair}(x, y)) \in B\}$$

□

Exercise 5. *State whether* $C = \{n^2 + n \mid \phi_n(n) = \mathit{undefined}\} \in \mathcal{RE}$*.*

Solution (sketch). $C \notin \mathcal{RE}$ since $\bar{\mathsf{K}} \leq_m C$ with reduction $h(n) = n^2 + n$. Such h is obviously computable and total. It is indeed a reduction because

- If $n \in \bar{\mathsf{K}}$ we have $\phi_n(n) =$ undefined, and so $h(n) = n^2 + n \in C$
- If $n \notin \bar{\mathsf{K}}$, we have that $\phi_n(n)$ is defined. Moreover, $h(n)$ is injective since it is strictly increasing. We can then have $h(n) \notin C$

□

Exercise 6. *Prove that* $\bar{\mathsf{K}} \leq_m \{i \mid \forall x.\ (x < 10 \implies \phi_i(x) = \phi_i(10))\} = D$*.*

Solution (sketch). The following is a reduction:

$$h(n) = \#\Big(\lambda x.\ \begin{cases} 3 & \text{if } x < 10 \land \phi_n(n) \text{ halts} \\ \mathit{undefined} & o.w. \end{cases}\Big)$$

the above is computable and total because "$x < 10$" is clearly recursive, while "$\phi_n(n)$ halts" is $\mathcal{RE}$. Overall, 1) the guard of the "if" is $\mathcal{RE}$, 2) the constant function of n "3" is recursive, and 3) in the other case the function is undefined.

The above is a reduction because:

- If $n \in \bar{\mathsf{K}}$, then $\phi_{h(n)}(x) = \begin{cases} 3 & \text{if } x < 10 \land \mathit{false} \\ \mathit{undefined} & o.w. \end{cases} = \mathit{undefined}$ for all x. In particular, $\phi_{h(n)}(x) = \mathit{undefined} = \phi_{h(n)}(10)$ for all x (including those $x < 10$). So, $h(n) \in D$.

- If $n \in \mathsf{K}$, then $\phi_{h(n)}(x) = \begin{cases} 3 & \text{if } x < 10 \\ undefined & o.w. \end{cases}$. In particular, $\phi_{h(n)}(5) = 3 \neq undefined = \phi_{h(n)}(10)$. Since $5 < 10$, we can conclude $h(n) \notin D$.

□

Exercise 7. *Let $A \otimes B = \{\mathsf{pair}(a, b) | a \in A \wedge b \in B\}$. Prove whether:*

$$A, B \in \mathcal{RE} \implies A \otimes B \in \mathcal{RE}$$
$$A \otimes B \in \mathcal{RE} \implies A \in \mathcal{RE}$$
$$A \otimes B \leq_m B \otimes A$$

(Suggestion: watch out for all the possible cases!)

Solution (sketch). Part 1. True. A semi-verifier can be constructed as follows: given the input x, we split it into $a = \mathsf{proj1}(x)$ and $b = \mathsf{proj2}(x)$. Then, we call both semi-verifiers $S_A(a)$ and $S_B(b)$. Formally:

$$S_{A \otimes B} = \lambda x.\ S_A(\mathbf{Proj1}\ x)(S_B(\mathbf{Proj2}\ x))$$

It is easy to check that $x \in A \otimes B$ iff $a \in A \wedge b \in B$. So, the above is indeed a semi-verifier for $A \otimes B$.

Part 2. False. $\bar{\mathsf{K}} \otimes \emptyset = \emptyset \in \mathcal{RE}$, but $\bar{\mathsf{K}} \notin \mathcal{RE}$.

Part 3. True. $h(x) = \mathsf{pair}(\mathsf{proj2}(x), \mathsf{proj1}(x))$ is a reduction. (Easy to check)

□

Exercise 8. *Prove whether $E = \{i | \forall x.\ \phi_i(x) = \phi_x(x)\} \in \mathcal{RE}$.*

Solution (sketch). $E \notin \mathcal{RE}$ by Rice-Shapiro ($\Rightarrow$). E is semantically closed (because...). The associated function set $\mathcal{F}_E$ only contains one function, namely $f(x) = \phi_x(x)$. The domain of this function is infinite, since it is defined e.g. on $\#(\mathbf{K}\ulcorner n \urcorner)$ for all n. Let g be any finite restriction of f. Since the domain of f is infinite, $g \neq f$. Hence $g \notin \mathcal{F}_E = \{f\}$. □

Exercise 9. $\star$ *Prove that: (note: $\mathcal{R}^t$ = recursive total functions)*

$$\forall f \in \mathcal{R}^t.\ \exists g \in \mathcal{R}^t.\ \forall i \in \mathbb{N}.\ \phi_{g(i)} = \phi_{f(g(i),i)}$$

Solution (sketch). Intentionally omitted. □

Exercise 10. $\star$ *Let $A_0, A_1, \ldots$ be an infinite sequence of sets of natural numbers. Let $f \in \mathcal{R}$ such that*

$$f(x, y) = \tilde{\chi}_{A_x}(y) = \begin{cases} 1 & \textit{if } y \in A_x \\ undefined & \textit{otherwise} \end{cases}$$

Prove that: (Remember to provide a justification)

[5% *score*]	$\forall x.\ A_x \in \mathcal{RE}$	*holds*
[25% *score*]	$\bigcup_x A_x \in \mathcal{RE}$	*holds*
[70% *score*]	$\bigcap_x A_x \in \mathcal{RE}$	*does not hold, in general*

Solution (sketch). Intentionally omitted. □