Computability Final Test — 2011-01-11

Definition 1. *The following exercises will refer to these subsets of* $\mathbb{N}$*:*

$A = \{i \mid \phi_i(4) \text{ is defined } \}$
$B = \{i \mid \phi_i(3) = \phi_i(5)\}$
$C = \{i \mid \phi_i(4) > 10\}$
$D = \{\#M \mid \forall x \in \mathbb{N}.\ M\ulcorner x \urcorner =_{\beta\eta} \ulcorner x^2 \urcorner\}$
$E = \{i \mid \forall x \in \mathbb{N}.\ \phi_i(2 \cdot x) + \phi_i(2 \cdot x + 1) = 4 \cdot x\}$
$F = \{i \mid \exists x \in \mathbb{N}.\ \big(\phi_i(x) = 7 \land \phi_i(x+1) \text{ not defined }\big)\}$
$G = \{i + 1 \mid \nexists a, b \in \mathbb{N}.\ a \neq 1 \land b \neq 1 \land i = a \cdot b\}$

Reminder*: when equating results of partial functions (as in* $\phi_i(3) = \phi_i(5)$ *above), we mean that* <u>*either*</u> *1) both sides of the equation are defined to be the same natural number,* <u>*or*</u> *2) both sides are undefined.*

Exercise 2. *Prove whether* D *is* λ*-definable. (Refer to Def. 1)*

Solution (sketch). No, by Rice (easy). □

Exercise 3. *Read the following argument, made by Mr. I. Forgot:*

> *Consider two arbitrary sets* $A, B \in \mathcal{RE}$*. From this, I know that each one is the range of some recursive partial function: accordingly, let* $f, g \in \mathcal{R}$ *such that* $A = \mathsf{ran}(f)$ *and* $B = \mathsf{ran}(g)$*. Then, I can construct the following partial function:*
>
> $$h(x) = \begin{cases} f(\mathsf{proj1}(x)) & \text{if } f(\mathsf{proj1}(x)) = g(\mathsf{proj2}(x)) \\ undefined & \text{otherwise} \end{cases}$$
>
> *The above* h *is recursive because... uhm... Anyway, we now prove that* $A \cup B$ *is ... uhm, was it* $A \cap B$*? ... uhm ...*

Question: what was Mr. Forgot trying to prove here? Write the statement of the property he was trying to prove. Then, help him by completing his proof.

Solution (sketch). He was trying to prove that $A \cap B \in \mathcal{RE}$. In fact, $\mathsf{ran}(h) = A \cap B$ is shown as follows.

- ($\mathsf{ran}(h) \supseteq A \cap B$) If $w \in A \cap B$, then we have $w \in \mathsf{ran}(f) \cap \mathsf{ran}(g)$, hence $w = f(y) = g(z)$ for some y, z. Hence, $h(\mathsf{pair}(y, z)) = w \in ran(h)$.
- ($\mathsf{ran}(h) \subseteq A \cap B$) If $w \in ran(h)$, then $w = h(x)$ for some x. Hence, $w = f(\mathsf{proj1}(x)) = g(\mathsf{proj2}(x))$ belongs to the ranges of f and g. So $w \in A \cap B$.

Finally, $h \in \mathcal{R}$ since it is equal to

$$l(x) = \begin{cases} f(\mathsf{proj1}(x)) & \text{if } f(\mathsf{proj1}(x)) = g(\mathsf{proj2}(x)) \text{ and both defined} \\ undefined & \text{otherwise} \end{cases}$$

It is easy to check that $h = l$ in all possible cases, including those in which $f(\mathsf{proj1}(x))$ or $g(\mathsf{proj2}(x))$ are undefined. Also, $l \in \mathcal{R}$ because we can semi-verify the property "$f(\mathsf{proj1}(x)) = g(\mathsf{proj2}(x))$ and both defined" by just running an implementation of f and g and comparing the results. Note that when f or g is undefined this procedure would loop, but in that case the property is false, so looping forever is indeed the correct behaviour for a semi-verifier.

We conclude that, since $A \cap B$ is the range of $h \in \mathcal{R}$, it is $\mathcal{RE}$. □

Exercise 4. *Prove whether $C \in \mathcal{R}$. (Refer to Def. 1)*
Then, prove whether $G \in \mathcal{R}$. (Refer to Def. 1)

Solution (sketch). For C: it is not recursive, as can be seen by Rice, since the set is semantically closed (depends on ϕ_i only), $\#(K\ulcorner 11\urcorner) \in C$, $\#(K\ulcorner 10\urcorner) \notin C$.

For G: we can see that $G = \{p + 1 \mid p \text{ prime}\}$. So to check whether $x \in G$ it suffices to check if ($x \neq 0$ and) $x - 1$ is prime. Deciding whether a natural is prime is of course possible, e.g. using the sieve of Eratosthenes. □

Exercise 5. *Prove whether $E \in \mathcal{RE}$. (Refer to Def. 1)*

Solution (sketch). E: no, by Rice-Shapiro ($\Rightarrow$). E is semantically closed (easy check), so let $\mathcal{E}$ its associated set of functions.

Let $f(x) = 2x$ when x is even, and zero otherwise. Clearly this is recursive, and $f \in \mathcal{E}$ since $f(2x) + f(2x + 1) = 4x + 0$.

Consider now any finite restriction g of f. We have $g(n)$ =undefined as soon as n is large enough. So, for large values of n, $g(2n) + g(2n + 1)$ is undefined, hence not $4n$. Therefore, $g \notin \mathcal{E}$. □

Exercise 6. *Prove whether $F \in \mathcal{RE}$. (Refer to Def. 1)*

Solution (sketch). F: no, by Rice-Shapiro ($\Leftarrow$). F is semantically closed (easy check), so let $\mathcal{F}$ its associated set of functions.

Consider the finite-domain function $g(0) = 7$ and undefined otherwise. We have $g \in \mathcal{F}$. Yet, its extension $f(x) = 7$ for all x, is always defined, so $f(x + 1) \neq$undefined for all x. Therefore $f \notin \mathcal{F}$. □

Exercise 7. *Prove that $A \leq_m C$. (Refer to Def. 1)*

Solution (sketch). We use the reduction

$$h(n) = \#\left(\lambda x.\ \phi_n(x) + 11\right)$$

and check that this indeed works. 1) When $n \in A$, $\phi_{h(n)}(4) = \phi_n(4) + 11 > 10$ since $\phi_n(4)$ is defined. Hence $h(n) \in C$. 2) When $n \notin A$, $\phi_{h(n)}(4) = \phi_n(4) + 11$ =undefined since $\phi_n(4)$ is undefined. Hence $h(4) \notin C$. □

Exercise 8. *Let $A \in \mathcal{RE}$. Prove that $B = \{n | \exists k \in \mathbb{N}.\mathsf{pair}(n,k) \in A\}$ is $\mathcal{RE}$.*

Solution (sketch). Let f be a recursive partial function s.t. $\mathsf{ran}(f) = A$. Then, the function $g(x) = \mathsf{proj1}(f(x))$ is a recursive partial function having range $\{\mathsf{proj1}(x) | x \in A\} = B$. $\square$

Exercise 9. *Prove whether there exists a sequence of sets $A_0, A_1, \ldots$ such that*

- $\bigcup_{i<k} A_i \notin \mathcal{RE}$ *for all* $k \in \mathbb{N}$
- $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{RE}$

Solution. Intentionally omitted. $\square$

Exercise 10. *Let $f(x,y)$ be a total recursive function, and let*

$$g(y) = \sum_{x=0}^{\infty} f(x,y) \textit{ whenever this is } < \infty\textit{; undefined otherwise}$$

Can we conclude that $g \in \mathcal{R}$? (Suggestion: consider a characteristic function)

Solution. Intentionally omitted. $\square$

Exercise 11. $\star$ *Prove that $\bar{\mathsf{K}} \leq_m B$. (Refer to Def. 1)*

Solution (sketch). Use the reduction

$$h(n) = \#\left(\lambda x.\ \begin{cases} \phi_n(n) & x = 3 \\ \text{undefined} & \text{otherwise} \end{cases}\right)$$

If $n \in \bar{\mathsf{K}}$, then $\phi_{h(n)}(3) = \phi_n(n) = \text{undefined} = \phi_{h(n)}(5)$, hence $h(n) \in B$. If $n \notin \bar{\mathsf{K}}$, then $\phi_{h(n)}(3) = \phi_n(n) \neq \text{undefined} = \phi_{h(n)}(5)$ hence $h(n) \notin B$. $\square$

Exercise 12. $\star$ *Prove whether the following partial function is recursive.*

$$f(n) = \max\Big(\{x+1 \mid \exists M.\ \#M \leq n\ \wedge\ M =_{\beta\eta} \ulcorner x \urcorner\}\Big)$$

Solution. Intentionally omitted. $\square$