Computability Midterm Test 1 — 2008-10-29

Exercise 1. *For each of the following λ-terms, state whether it has a $\beta\eta$-normal form. Justify your answer.*

$$\mathbf{\Theta}(\mathbf{KI}), \quad \mathbf{KI}\Omega, \quad \ulcorner 2 \urcorner \ulcorner 2 \urcorner, \quad \mathbf{Mul}\,\mathbf{0}\,\Omega$$

Answer.

- Yes, $\mathbf{\Theta}(\mathbf{KI}) =_{\beta\eta} \mathbf{KI}(\mathbf{\Theta}(\mathbf{KI})) =_{\beta\eta} \mathbf{I} \not\to_{\beta\eta}$
- Yes, $\mathbf{KI}\Omega =_{\beta\eta} \mathbf{I} \not\to_{\beta\eta}$
- Yes, $\ulcorner 2 \urcorner \ulcorner 2 \urcorner =_{\beta\eta} (\lambda sz.\, s(sz))\ulcorner 2 \urcorner =_{\beta\eta} \lambda z.\, \ulcorner 2 \urcorner(\ulcorner 2 \urcorner z) =_{\beta\eta} \lambda z.\, (\lambda sz.\, s(sz))(\ulcorner 2 \urcorner z) =_{\beta\eta}$ $\lambda z.\, (\lambda \bar{z}.\, (\ulcorner 2 \urcorner z)((\ulcorner 2 \urcorner z)\bar{z})) =_{\beta\eta} \lambda z\bar{z}.\, \ulcorner 2 \urcorner z(\ulcorner 2 \urcorner z\bar{z}) =_{\beta\eta} \lambda z\bar{z}.\, \ulcorner 2 \urcorner z(z(z\bar{z})) =_{\beta\eta}$ $\lambda z\bar{z}.\, z(z(z(z\bar{z}))) =_{\beta\eta} \lambda sz.\, s(s(s(sz))) =_{\beta\eta} \ulcorner 4 \urcorner \not\to_{\beta\eta}$
- Yes, $\mathbf{Mul}\,\mathbf{0}\,\Omega =_{\beta\eta} \mathbf{0}(\mathbf{Add}\Omega)\mathbf{0} =_{\beta\eta} \mathbf{0} \not\to_{\beta\eta}$

Exercise 2. *Compute the natural number $\#(\lambda x_0.\lambda x_1.x_1)$. Then, define M such that $\#M = 224$. Then, define N such that $\#N = 49$.*

Answer.

- $\#(\lambda x_0.\lambda x_1.x_1) = \mathsf{inR}(\mathsf{inR}(\mathsf{pair}(0, \#(\lambda x_1.x_1))))$ and $\#(\lambda x_1.x_1) = \mathsf{inR}(\mathsf{inR}(\mathsf{pair}(1, \#(x_1))))$ and $\#x_1 = \mathsf{inL}(1) = 2$. Unfolding the definitions, we get 1987.
- 224 is even: $224 = \mathsf{inL}(112)$. So, $M = x_{112}$.
- 49 is odd: $49 = \mathsf{inR}(24)$. 24 is even: $24 = \mathsf{inL}(12)$. So, N is an application N_1N_2. By a table lookup, $12 = \mathsf{pair}(2, 2)$. So, $\#N_1 = \#N_2 = 2$. 2 is even: $2 = \mathsf{inL}(1)$. So, $N = x_1x_1$.

Exercise 3. *State whether these functions and sets are λ-definable, and justify your answer.*

$$f(n) = \prod_{i=0}^{n-1}(i^2 + 1)$$

$$A = \{\#M | (\lambda x.M) =_{\beta\eta} \mathbf{I}\} \qquad B = \{\#M | \exists N.\, \#M = \#(NN)\}$$

$$C = \{\#M | M =_{\beta\eta} \mathbf{K}M\} \qquad D = \{\#M + 1 | M =_{\beta\eta} \ulcorner M \urcorner\, \mathbf{Pred}\, \ulcorner M \urcorner\}$$

$$g(n) = \begin{cases} 2 \cdot n & \textit{when } n = \#M \textit{ and } M =_{\beta\eta} \mathbf{I} \\ 5 & \textit{otherwise} \end{cases}$$

Answer.

- $f(0) = 1$ and $f(n) = ((n-1)^2+1) \cdot f(n-1)$ otherwise. So f is λ-defined by

  $F = \mathbf{\Theta}(\lambda f n.\, \mathbf{Eq} n \mathbf{0} \ulcorner 1 \urcorner (\mathbf{Mul}(\mathbf{Succ}(\mathbf{Mul}(\mathbf{Pred}\, n)(\mathbf{Pred}\, n)))(f(\mathbf{Pred}\, n)))$

- We apply Rice's Theorem to A. Clearly, $\#x \in A$, so $A \neq \emptyset$. Also, $\#y \notin A$, so $A \neq \mathbb{N}$. Then, assuming $\#M \in A$ and $M =_{\beta\eta} N$, we get $\mathbf{I} = \lambda x.\, M =_{\beta\eta} \lambda.\, N$, so also $\#N \in A$, and A is closed under $\beta\eta$. Result: A is not λ-definable.

- B is λ-defined by

  $V_B = \lambda m.\, \mathbf{Case}\, m\, (\mathbf{KF})(\lambda y.\, \mathbf{Case}\, y\, (\lambda z.\, \mathbf{Eq}(\mathbf{Proj1}\, z)(\mathbf{Proj2}\, z))(\mathbf{KF}))$

  (2010 note: this is now done using **Sd** in a simpler way.)

- C is not λ-definable: we apply Rice's Theorem. $\#(\mathbf{\Theta K}) \in C$ by the fundamental fixed point property. $\#\mathbf{I} \notin C$, otherwise $\mathbf{I} =_{\beta\eta} \mathbf{KI} =_{\beta\eta} \lambda xy.y$ and the latter is a normal form distinct from $\mathbf{I}$, so $\neq_{\beta\eta}$. So, $\emptyset \neq C \neq \mathbb{N}$. To show C closed under $\beta\eta$, take $\#M \in C$ and $M =_{\beta\eta} N$, we then get $N =_{\beta\eta} M =_{\beta\eta} \mathbf{K}M =_{\beta\eta} \mathbf{K}N$, so $\#N \in C$. This concludes.

- D is not λ-definable. We first note that $\ulcorner n \urcorner \mathbf{Pred} \ulcorner n \urcorner = \mathbf{0}$, for any n. So, we get $D = \{\#M + 1 | M =_{\beta\eta} \mathbf{0}\}$. By contradiction, assume D to be λ-defined by V_D. Then, $V_E = \lambda m.\, V_D(\mathbf{Succ}\, m)$ proves that $E = \{\#M | M =_{\beta\eta} \mathbf{0}\}$ is λ-definable: if $n \in E$, then $n+1 \in D$ and $V_E \ulcorner n \urcorner = V_D \ulcorner n+1 \urcorner = \mathbf{T}$; otherwise if $n \notin E$, then $n+1 \notin D$ and $V_E \ulcorner n \urcorner = V_D \ulcorner n+1 \urcorner = \mathbf{F}$. We get a contradiction by showing that E is actually λ-undefinable. By Rice, $\#\mathbf{0} \in E$ and $\#\ulcorner 1 \urcorner \notin E$. Also, taking $\#M \in E$ and $M =_{\beta\eta} N$, we then get $N =_{\beta\eta} M =_{\beta\eta} \mathbf{0}$, so $\#N \in E$. This concludes.

- g is not λ-definable. By contradiction, assume G defines it. Then the set $A = \{\#M | M =_{\beta\eta} \mathbf{I}\}$ can be λ-defined by $V_A = \lambda m.\, \mathbf{Even}(Gm)$. Indeed, if $n \in A$, then $g(n) = 2 \cdot n$ and $V_A \ulcorner n \urcorner = \mathbf{T}$; otherwise if $n \notin A$, then $g(n) = 5$ and $V_A \ulcorner n \urcorner = \mathbf{F}$. We get a contradiction by showing that A is actually λ-undefinable. By Rice, $\#I \in A$ and $\#\Omega \notin A$. Also, taking $\#M \in A$ and $M =_{\beta\eta} N$, we then get $N =_{\beta\eta} M =_{\beta\eta} \mathbf{I}$, so $\#N \in A$. This concludes.

Exercise 4. *Prove or refute the following statements.*

- *A is λ-definable if and only if $A \setminus \{5\}$ is λ-definable.*

- *If A and B are* not *λ-definable, then $A \cup B$ is* not *λ-definable.*

- *If A is* not *λ-definable and $A \subseteq B$, then B is* not *λ-definable.*

Answer.

- The first point holds. ($\Rightarrow$) If A is defined by V_A, then $V_{A\setminus\{5\}} = \lambda n.\, \mathbf{Eq}\, n \ulcorner\ulcorner 5 \urcorner\urcorner \mathbf{F}\, (V_A n)$ defines $A \setminus \{5\}$ (trivial check).

  ($\Leftarrow$) If $A \setminus \{5\}$ is defined by $V_{A\setminus\{5\}}$, then we consider two cases. If $5 \notin A$, then $A \setminus \{5\} = A$ and choosing $V_A = V_{A\setminus\{5\}}$ is enough to λ-define A (trivial check). Otherwise, if $5 \in A$, we choose $V_A = \lambda n.\, \mathbf{Eq}\, n \ulcorner\ulcorner 5 \urcorner\urcorner \mathbf{T}\, (V_{A\setminus\{5\}} n)$, and this λ-defines A (trivial check).

- The second point does not hold, in general. Take A to be any λ-undefinable set (e.g. A from Exercise 3). Pick $B = \mathbb{N} \setminus A$. We have that B is not λ-definable: otherwise $V_A = \lambda n.\, \mathbf{Not}\, (V_B n)$ defines A. However, $A \cup B = \mathbb{N}$ which is trivially λ-defined by $\mathbf{KT}$.

- The third point does not hold, in general. Take A to be any λ-undefinable set (e.g. A from Exercise 3). Pick $B = \mathbb{N}$. Clearly, $A \subseteq B$, but B is trivially λ-defined by $\mathbf{KT}$.

Exercise 5. *Define λ-terms $M_1, \ldots, M_5$ such that*

$$
\begin{array}{ll}
\forall N. & M_1 \ulcorner N \urcorner =_{\beta\eta} \ulcorner N N \mathbf{K} \urcorner \\
\forall N, i. & M_2 \ulcorner \lambda x_i.\, N \urcorner =_{\beta\eta} \ulcorner \lambda x_{33}.\, N N \urcorner \\
\forall N. & M_3 \ulcorner N \urcorner =_{\beta\eta} \ulcorner \lambda x_{\# N}.\, N \urcorner \\
\forall N. & M_4 \ulcorner N \urcorner =_{\beta\eta} \ulcorner N \ulcorner M_4 \urcorner \urcorner \\
\forall N \in \Lambda^0. & M_5 \ulcorner N \urcorner =_{\beta\eta} N \ulcorner\ulcorner \# N + 1 \urcorner\urcorner
\end{array}
$$

Answer.

(2010 note: all of these can now be done using $\mathbf{Sd}, \mathbf{Var}, \mathbf{App}, \mathbf{Lam}$ in a simpler way.)

- $M_1 = \lambda n.\, \mathbf{App}(\mathbf{App}\, n\, n) \ulcorner \mathbf{K} \urcorner$
- $M_2 = \lambda n.\, \mathbf{Case}\, n\, \Omega\, (\lambda y.\, \mathbf{Case}\, y\, \Omega\, (\lambda z.\, \mathbf{InR}(\mathbf{InR}(\mathbf{Pair} \ulcorner\ulcorner 33 \urcorner\urcorner N))))$
  $N = \mathbf{InR}(\mathbf{InL}(\mathbf{Pair}(\mathbf{Proj2}\, z)(\mathbf{Proj2}\, z)))$
- $M_3 = \lambda n.\, \mathbf{InR}(\mathbf{InR}\, (\mathbf{Pair}\, n\, n))$
- We rewrite the question as
  $M_4 \ulcorner N \urcorner = \mathbf{App} \ulcorner N \urcorner \ulcorner\ulcorner M_4 \urcorner\urcorner$, which is
  $M_4 \ulcorner N \urcorner = \mathbf{App} \ulcorner N \urcorner (\mathbf{Num} \ulcorner M_4 \urcorner)$, which is
  $M_4 = (\lambda mn.\, \mathbf{App}\, n\, (\mathbf{Num}\, m)) \ulcorner M_4 \urcorner$.
  The above is not yet a proper definition, but such an M_4 can be constructed through the second fixed point theorem. To be precise,
  $M_4 = M \ulcorner M \urcorner$, where
  $M = \lambda w.\, F(\mathbf{App}\, w\, (\mathbf{Num}\, w))$
  $F = \lambda mn.\, \mathbf{App}\, n\, (\mathbf{Num}\, m)$.

- $M_5 = \lambda n.\, \mathbf{E}\, n\, (\mathbf{Succ}\, n)$
  where **E** is the universal program.