Computability Final Test — 2009-06-03

Part 1

Exercise 1. *State whether these sets are λ-definable.*

$$
\begin{aligned}
A &= \{\#M \mid M =_{\beta\eta} \mathbf{S}M\}\\
B &= \{\#M + 1 \mid M\Omega =_{\beta\eta} M\mathbf{I}\}\\
C &= \{\#M \mid \exists N.\, NM =_{\beta\eta} \mathbf{I}\}\\
D &= \{\#M \mid \forall N.\, NM =_{\beta\eta} \mathbf{I}\}\\
E &= \{\#M \mid \mathbf{E}\ulcorner M\urcorner =_{\beta\eta} \ulcorner\ulcorner 4\urcorner\urcorner\}\\
F &= \{\#M \mid \forall N.\, MN =_{\beta\eta} \ulcorner N\urcorner\}\\
G &= \{\#M \mid \exists N.\, MN =_{\beta\eta} \ulcorner N\urcorner\}\\
H &= \{\#M \mid \mathbf{Even}\,(\mathbf{Mul}\,\ulcorner M\urcorner\ulcorner M\urcorner) =_{\beta\eta} \mathbf{T}\}
\end{aligned}
$$

Answer. (sketch)

- By Rice. We have $\#(\mathbf{\Theta}S) \in A$, $\Omega \notin A$. If $\#M \in A$ and $M = N$, then from $M = \mathbf{S}M$ we get $N = \mathbf{S}N$, so $\#N \in A$. So A is not λ-def.
- By contradiction, assume that V_B is a verifier. Then, $V = \lambda n.\, V_B(\mathbf{Succ}n)$ is a verifier for $B' = \{\#M \mid M\Omega =_{\beta\eta} M\mathbf{I}\}$, which is not λ-def., as can be seen by Rice (easy to check).
- For any M, if we take $N = \mathbf{KI}$, we have $NM = \mathbf{I}$. So $C = \mathbb{N}$ and is λ-def.
- For any M, if we take $N = \Omega$, we have $\Omega M \neq \mathbf{I}$, since ΩM has no normal form (since Ω is unsolvable), while $\mathbf{I}$ has. So $D = \emptyset$ is λ-def.
- A not completely correct (yet acceptable) answer: $\mathbf{E}\ulcorner M\urcorner = M$, so $E = \{\#M \mid M =_{\beta\eta} \ulcorner\ulcorner 4\urcorner\urcorner\}$, which is not λ-def. by Rice (easy to check). The formally correct answer: $E = \{\#M \mid M\{\Omega/\mathsf{free}(M)\} =_{\beta\eta} \ulcorner\ulcorner 4\urcorner\urcorner\}$, and to apply Rice you have to show that $M\{\Omega/\mathsf{free}(M)\} =_{\beta\eta} N\{\Omega/\mathsf{free}(N)\}$ whenever $M =_{\beta\eta} N$.
- $F = \emptyset$, otherwise $M\mathbf{I} = \ulcorner\mathbf{I}\urcorner \neq \ulcorner\mathbf{II}\urcorner = M(\mathbf{II}) = M\mathbf{I}$ (contradiction). So F is λ-def.

- $\#\Omega \notin G$, since ΩN can not have a normal form. $\#(\mathbf{K}\ulcorner\mathbf{I}\urcorner) \in G$ since we can take $N = \mathbf{I}$. If $M = O$, and $\#M \in G$, then $MN = \ulcorner N \urcorner$ for some N, and $ON = \ulcorner N \urcorner$ (for the same N), so $\#O \in G$. By Rice, G is not λ-def.

- $V_H = \lambda n.\mathbf{Even}(\mathbf{Mul}\, n\, n)$ is a verifier, so H is λ-def. □

Exercise 2. *Find $M, N \in \Lambda^0$ such that*

$$M\mathbf{I}N \neq_{\beta\eta} N\mathbf{I}M$$

Answer. Take $M = \mathbf{F}$ and $N = \mathbf{I}$. Then, $M\mathbf{I}N = \mathbf{FII} = \mathbf{I}$ while $\mathbf{IIF} = \mathbf{F}$. Since $\mathbf{I}$ and $\mathbf{F}$ are distinct normal forms, we conclude. □

Exercise 3. *Define $M \in \Lambda^0$ such that*

$$\begin{aligned} &M\ulcorner \lambda x_i.\, N\urcorner = M\ulcorner NN\urcorner \\ &M\ulcorner NO\urcorner = M\ulcorner O\urcorner \\ &M\ulcorner x_i\urcorner = \ulcorner\ulcorner i+2\urcorner\urcorner \end{aligned}$$

Answer.

$$\begin{aligned} M =& \mathbf{\Theta}(\lambda gn.\mathbf{Case}\, n\, (\mathbf{Add}\ulcorner\ulcorner 2\urcorner\urcorner)\, A) \\ A =& \lambda y.\, \mathbf{Case}\, y\, B\, C \\ B =& \lambda z.\, g(\mathbf{Proj2}\, z) \\ C =& \lambda z.\, g(\mathbf{App}(\mathbf{Proj2}\, z)(\mathbf{Proj2}\, z)) \end{aligned}$$

(2010 note: this can now be solved using **Sd** in a simpler way.) □

Exercise 4. *Prove of refute the following:*

$$MMMM =_{\beta\eta} MM \implies MM =_{\beta\eta} M$$

Answer. We refute it by taking $M = \mathbf{K}$. We have $MMMM = \mathbf{KKKK} = (\mathbf{KKK})\mathbf{K} = \mathbf{KK} = MM$. However $MM = \mathbf{KK} = \lambda x.\mathbf{K} = \lambda xyz.y$ which is a normal form different from $\mathbf{K} = \lambda xy.\, x$, so $MM \neq M$. □

Part 2

Exercise 5. *Let $f \in (\mathbb{N} \to \mathbb{N})$. Assume that f is total, and that for all i, j, whenever $\phi_i = \phi_j$, we have $f(i) = f(j)$. State whether, under these assumptions, we can conclude any of the following:*

- *f must be computable*

- *f may be computable, but may as well be non computable*

- *f must be non computable*

Further, assume that $\mathsf{ran}(f)$ is a finite set with an even number of elements. Does the answer to the above change?

Answer. The function f may be computable (take $f(x) = 0$), and may be non computable, e.g. $f = \chi_{\mathsf{K}_0}$, the characteristic of

$$\mathsf{K}_0 = \{n | \phi_n(0) \text{ is defined}\}$$

If we assume that $\mathsf{ran}(f)$ is a finite set with an even cardinality, then f must be non computable. This is because, since f is total, we have $f(3) \in \mathsf{ran}(f)$, so $\mathsf{ran}(f)$ is not empty. Since 1 is odd, $\mathsf{ran}(f)$ can not be a singleton, so we have $x, y \in \mathsf{ran}(f)$ for some distinct naturals x, y, that is $x = f(a) \neq f(b) = y$ for some a and b. We can then consider

$$A = \{n | f(n) = x\}$$

The set is semantically closed since f returns the same value on indexes of equivalent programs. The set is not empty since $a \in A$. The set is not $\mathbb{N}$ since $b \notin A$. By Rice, A is not recursive. However, is f were recursive, we could use it on n and check the result against x to verify $n \in A$. Contradiction. □

Exercise 6. *State whether these functions are computable.*

$$f(n) = \begin{cases} x & \text{if } x \text{ is the least } x \text{ such that } \phi_n(x) = 3 \\ 0 & \text{if no such } x \text{ exists} \end{cases}$$
$$g(n) = \phi_{\phi_n(n+1)}(n+1)$$

Answer. (sketch)

- By contradiction, assume $f \in \mathcal{R}$. Then take

  $$A = \{n | f(n) = 1\} = \{n | \phi_n(1) = 3 \wedge \phi_n(0) \text{ is not defined or } \neq 3 \ \}$$

  Since $f \in \mathcal{R}$, $A \in \mathcal{R}$ since we can write a verifier for A computing f and checking its result against 1. However, its is easy to show that $A \notin \mathcal{R}$ using Rice.

- Take the universal function $u(x, y) = \phi_x(y)$. Then $g(n) = u(u(n, n + 1), n + 1)$ is a composition of u and $+$, so $g \in \mathcal{R}$.

□

Exercise 7. *Define two total functions $f, g \in (\mathbb{N} \to \mathbb{N})$ such that $(f \circ g) \in \mathcal{R}$, $f \notin \mathcal{R}$, and $g \notin \mathcal{R}$.*

Answer. Take

$$g(n) = \chi_{\mathsf{K}}(n) \qquad\qquad f(n) = \begin{cases} 0 & \text{if } n < 2 \\ \chi_{\mathsf{K}}(n-2) & \text{otherwise} \end{cases}$$

We have $g \notin \mathcal{R}$ because $\mathsf{K} \notin \mathcal{R}$. We have $g \notin \mathcal{R}$ because otherwise $\chi_{\mathsf{K}}(n)$ could be computed using $g(n+2)$. For the composition we have that $f(g(n))$ is either $f(0)$ or $f(1)$, depending on whether $n \in \mathsf{K}$. In both cases, $f(g(n)) = 0$. So $f \circ g$ is the constant null function, which is recursive. □

Exercise 8. *State whether these sets belong to $\mathcal{R}, \mathcal{RE} \setminus \mathcal{R}$, or neither.*

$$\begin{aligned}
A &= \{\mathsf{pair}(n,m) | \forall x.\, (\ \phi_n(x) \ \textit{and}\ \phi_m(x)\ \textit{defined}\ \wedge \phi_n(x) < \phi_m(x)\ )\} \\
B &= \{\mathsf{pair}(n,m) | \exists x.\, (\ \phi_n(x) \ \textit{and}\ \phi_m(x)\ \textit{defined}\ \wedge \phi_n(x) < \phi_m(x)\ )\} \\
C &= \{\phi_n(n) | n > 1000 \wedge \phi_n(n)\ \textit{defined}\} \\
D &= \{\phi_n(n) | n < 1000 \wedge \phi_n(n)\ \textit{defined}\} \\
E &= \{k | \exists n > k.\, \phi_n(n)\ \textit{does not halt in } k \textit{ steps}\} \\
F &= \{n | \exists k.\, \phi_n(k) = \phi_n(k+1) \wedge\ \textit{both defined}\ \} \\
G &= \{n | \exists k \forall x.\, \phi_n(x) = k\}
\end{aligned}$$

Answer.

- $A \notin \mathcal{RE}$. By contradiction, assume that S_A is a semi-verifier for A. Then, $S_{A'} = \lambda n.\, V_A(\mathbf{Pair}\, n \ulcorner \mathbf{K} \ulcorner 1 \urcorner\urcorner)$ is a semi-verifier for the set

  $$A' = \{n | \forall x.\, \phi_n(x) \text{defined} \wedge \phi_n(x) < 1\} = \{n | \forall x.\, \phi_n(x) = 0\}$$

  That is, $A' = \{n | \phi_n \in \mathcal{F}\}$ where $\mathcal{F} = \{f\}$ and f is the constant null function. By Rice-Shapiro ($\Rightarrow$), we have that some finite restriction g of f must belong to $\mathcal{F}$, but $\mathcal{F}$ only contains the f which has infinite domain.

- The predicate

  $$p(x,n,m,k,j) = \phi_n(x) \text{ halts in } k \text{ steps} \wedge \phi_m(x) \text{ halts in } i \text{ steps} \wedge \phi_n(x) < \phi_n(x)$$

  is recursive, since we only need to run programs for a bounded number of steps. This means that the predicate

  $$q(n,m) = \exists x, k, i.\, p(x,n,m,k,j)$$

  is $\mathcal{RE}$, and we have a semi-verifier S_q for it. Then, $S_B = \lambda y.\, S_q\, (\mathbf{Proj1}\, y)\, (\mathbf{Proj2}\, y)$ shows that $B \in \mathcal{RE}$.

  We also have $B \notin \mathcal{R}$, otherwise from V_B we can construct $V_{B'} = \lambda n.\, V_B(\mathbf{Pair}\, n \ulcorner \mathbf{K} \ulcorner 1 \urcorner\urcorner)$ for the set $B' = \{n | \exists x.\, \phi_n(x) = 0\}$ which is not recursive, as we can see using Rice: $\#(\mathbf{K} \ulcorner 0 \urcorner) \in B'$, $\#(\mathbf{K} \ulcorner 1 \urcorner) \notin B'$, and B' depends only on ϕ_n so it is semantically closed.

- $C = \mathbb{N} \in \mathcal{R}$. To prove that $x \in C$ for all x, take $i = \mathsf{pad}(\mathsf{pad}(\cdots \#(\mathbf{K}\ulcorner x \urcorner)))$ where pad is applied 1000 times. Clearly, $i > 1000$ and $\phi_i(i) = x$.

- D is finite (contains at most 1000 elements), and so $\mathcal{R}$.

- Given any k, take $n = \mathsf{pad}(\mathsf{pad}(\cdots \#\Omega))$ where pad is applied $k+1$ times. This ensures that $n > k$ and $\phi_n(n)$ never halts. Since we can always find n for any k, the condition expressed in set E is always true. So $E = \mathbb{N} \in \mathcal{R}$.

- $F \in \mathcal{RE}$. Indeed, the predicate

  $$p(n, k, i, j) = \phi_n(k) \text{ halts in } i \text{ steps} \wedge \phi_n(k{+}1) \text{ halts in } j \text{ steps} \wedge \phi_n(k) = \phi_n(k{+}1)$$

  is $\mathcal{R}$ since we run programs for only a bounded number of steps. so, the predicate $q(n) = \exists k, i, j.\, p(n, k, i, j)$ is $\mathcal{RE}$.

  The set E is not recursive, as we can see using Rice. Clearly $\#(\mathbf{K}\ulcorner 5 \urcorner) \in F$, while $\#\mathbf{I} \notin F$. The set depends only on ϕ_n, so it is semantically closed.

- $G \notin \mathcal{RE}$. Otherwise we apply Rice-Shapiro ($\Rightarrow$) to

  $$\mathcal{F} = \{f \in \mathcal{R} | f \text{ is a constant function}\}$$

  If we take $f(n) = 0$ we have $f \in \mathcal{F}$, but no finite restriction of f belongs to $\mathcal{F}$. $\square$