Computability Final Test — 2009-02-06

Part 1

Exercise 1. *Write a λ-term defining the following function.*

$$f(x,y) = \begin{cases} x - y & \text{if } x > y \\ 3 & \text{otherwise} \end{cases}$$

Answer. $\lambda xy.\, \mathbf{Lt}\, y\, x (\mathbf{Sub}\, x\, y) \ulcorner\!\ulcorner 3 \urcorner\!\urcorner$

Exercise 2. *Construct λ-terms M_0, M_1 such that:*

$$\begin{array}{l} M_0 = \ulcorner M_0 \urcorner \\ M_1 \ulcorner \lambda x_i.\, N \urcorner = \ulcorner\!\ulcorner i \urcorner\!\urcorner \\ M_1 \ulcorner x_i \urcorner = \ulcorner\!\ulcorner i \urcorner\!\urcorner \\ M_1 \ulcorner N_0 N_1 \urcorner = M_1 \ulcorner N_0 \urcorner \end{array}$$

for all λ-terms N, N_0, N_1 and natural i.

Answer. M_0 is from the second recursion theorem applied on $\mathbf{I}$.

$$M_0 = M \ulcorner M \urcorner \qquad M = \lambda w.\, \mathbf{I}(\mathbf{App}\, w (\mathbf{Num}\, w))$$

M_1 requires recursion:

$$\begin{array}{l} M_1 = \mathbf{\Theta}(\lambda gn.\, \mathbf{Case}\, n\, \mathbf{I} A) \\ A = \lambda y.\, \mathbf{Case}\, y\, B\, \mathbf{Proj1} \\ B = \lambda z.\, g(\mathbf{Proj1}\, z) \end{array}$$

(2010 note: now this can be solved using **Sd** in a more direct fashion.)

Exercise 3. *State whether these sets are λ-definable. Justify your answers.*

$$\begin{array}{l} A = \{\#M | M =_{\beta\eta} \mathbf{\Theta}(\mathbf{K} M)\} \\ B = \{\#M | \forall n \in \mathbb{N}.\, M \ulcorner\!\ulcorner n + 5 \urcorner\!\urcorner =_{\beta\eta} \mathbf{T}\} \\ C = \{\#(\lambda x_i.\, M) | M =_{\beta\eta} \ulcorner\!\ulcorner i \urcorner\!\urcorner\} \\ D = \{\#M | \exists n \in \mathbb{N}.\, \mathbf{Add} \ulcorner M \urcorner \ulcorner\!\ulcorner n \urcorner\!\urcorner =_{\beta\eta} \ulcorner\ulcorner M \urcorner\urcorner\} \\ E = \{\#M | MM =_{\beta\eta} MMM\} \end{array}$$

Answer (sketch).

- Since $\mathbf{\Theta}(\mathbf{K}M) = \mathbf{K}M(\mathbf{\Theta}(\mathbf{K}M)) = M$, $A = \{\#M | M =_{\beta\eta} \mathbf{\Theta}(\mathbf{K}M)\} = \Lambda$ which is λ-defined by $\mathbf{KT}$.

- B is not λ-definable, since Rice's theorem applies. $\#(\mathbf{KT})$ belongs to B, while $\#\Omega$ does not. B is trivially closed under $\beta\eta$.

- C is not λ-definable. By contradiction, let V_C be a verifier. Then, we can build a verifier $V_{C'}$ for $C' = \{\#M | M =_{\beta\eta} \ulcorner 0 \urcorner\}$ which is not λ-definable, as can be seen through Rice's theorem. Here is how to construct $V_{C'}$:

  $$V_{C'} = \lambda n.\, V_C(\mathbf{InR}\,(\mathbf{InR}\,(\mathbf{Pair}\,\ulcorner 0 \urcorner\, n)))$$

  (This can also be justified through the $\leq_m$ relation.)

  (2010 note: now we would simply use **Lam** above.)

- D is λ-definable, since

  $$\begin{aligned} D &= \{m | \exists n \in \mathbb{N}.\, \mathbf{Add}\,\ulcorner m \urcorner \ulcorner n \urcorner =_{\beta\eta} \ulcorner\ulcorner m \urcorner\urcorner\} \\ &= \{m | \exists n \in \mathbb{N}.\, \ulcorner m + n \urcorner =_{\beta\eta} \ulcorner \#\ulcorner m \urcorner\urcorner\} \\ &= \{m | \exists n \in \mathbb{N}.\, m + n = \#\ulcorner m \urcorner\} \\ &= \{m | m \leq \mathsf{num}(m)\} \end{aligned}$$

  and the last condition is trivial to check.

- E is not λ-definable. By Rice: E is clearly closed under $\beta\eta$. $\#\mathbf{I}$ trivially belongs to E. To check that $\#\mathbf{K}$ does not belong to E, we proceed by contradiction: if $\mathbf{KK} = \mathbf{KKK}$, then $\mathbf{KK} = \mathbf{K}$, hence $(\lambda xyz.y) =_{\beta\eta} (\lambda yz.y)$, which is impossible since both are normal forms.

Exercise 4. *State whether, for all $G \in \Lambda^0$, there exists some $M \in \Lambda^0$ such that $M = GMG\ulcorner M\, G\, M \urcorner$.*

Answer (sketch). Yes. First rewrite the equation as $M = FM\ulcorner M \urcorner$ for a suitable F. Then apply $\mathbf{\Theta}$ to remove the recursion, transforming the equation in $M = F'\ulcorner M \urcorner$. Finally, apply the second recursion theorem.

Part 2

Exercise 5. *State whether the following functions belong to $\mathcal{R}$. Justify your* *answers.*

$$f(n) = \begin{cases} \phi_n(3) & \textit{if } \phi_n(3) \textit{ is even} \\ \phi_n(3) + 11 & \textit{if } \phi_n(3) \textit{ is odd} \\ 8 & \textit{if } \phi_n(3) \textit{ is undefined} \end{cases}$$
$$g(n) = \begin{cases} \phi_n(3) & \textit{if } \phi_n(3) \textit{ is even} \\ \phi_n(3) + 11 & \textit{if } \phi_n(3) \textit{ is odd} \\ 18 & \textit{if } \phi_n(3) \textit{ is undefined} \end{cases}$$
$$h(n) = \begin{cases} undefined & \textit{if } \phi_n(3+n) \textit{ is defined} \\ 3 + n & \textit{otherwise} \end{cases}$$

Answer (sketch).

- $f \notin \mathcal{R}$. Otherwise, by checking whether $f(n) = 8$, we could decide the set
$$\{n | \phi_n(3) = 8 \lor \phi_n(3) = \text{undefined } \}$$
which is not recursive, as can be seen through Rice.

- $g \notin \mathcal{R}$. Otherwise, by checking whether $f(n) = 18$, we could decide the set
$$\{n | \phi_n(3) = 18 \lor \phi_n(3) = \text{undefined} \lor \phi_n(3) = 7 \ \}$$
which is not recursive, as can be seen through Rice.

- $h \notin \mathcal{R}$. Otherwise, consider the following total recursive function:
$$a(n) = \#(\lambda x.\, \phi_n(n))$$
We have that $b(n) = h(a(n))$ is recursive, and defined whenever $\phi_{a(n)}(a(n)+3)$ is not defined, i.e. whenever $(\lambda x.\, \phi_n(n))\ulcorner a(n) + 3 \urcorner$ does not evaluate to a numeral, i.e. when $\phi_n(n)$ is not defined. So, $\mathsf{dom}(b) = \bar{\mathsf{K}}$, hence $\bar{\mathsf{K}} \in \mathcal{RE}$. Contradiction.

Exercise 6. *State whether the following sets belong to either $\mathcal{R}$, $\mathcal{RE} \setminus \mathcal{R}$, or they do not belong to $\mathcal{RE}$. Justify your answers.*

$A = \{n | \phi_n \subseteq \mathsf{id}\}$ *where* $\mathsf{id} \in (\mathbb{N} \to \mathbb{N})$ *is the identity function*
$B = \{n | \forall x.\, \phi_n(2 \cdot x) \textit{ is defined}\}$
$C = \{n | \forall x.\, \phi_n(2 \cdot x + 1) \textit{ is not defined}\}$
$D = \{\mathsf{pair}(\max(1000, k), n) | \phi_n(n) \textit{ halts in } k \textit{ steps}\}$
$E = \{\mathsf{pair}(\min(1000, k), n) | \phi_n(n) \textit{ halts in } k \textit{ steps}\}$

Answer.

- $A \notin \mathcal{RE}$, by Rice-Shapiro. Indeed, otherwise, we let $\mathcal{F} = \{f \in \mathcal{R} | f \subseteq \mathsf{id}\}$. Clearly, $A = \{n | \phi_n \in \mathcal{F}\}$. We have that the always undefined function $f_\emptyset$ belongs to $\mathcal{F}$. By Rice-Shapiro (direction: $\Leftarrow$), since $\mathsf{dom}(f_\emptyset) = \emptyset$ is finite, any computable super-function of $f_\emptyset$ belongs to $\mathcal{F}$. So, $g(n) = 5$ belongs to $\mathcal{F}$. Contradiction, since $g \subseteq \mathsf{id}$ is false.

- $A \notin \mathcal{RE}$, by Rice-Shapiro. Indeed, otherwise, we let $\mathcal{F} = \{f \in \mathcal{R} | \forall x.\, f(2 \cdot x) \text{ is defined}\}$. Clearly, $B = \{n | \phi_n \in \mathcal{F}\}$. We have that the always zero function $f(n) = 0$ belongs to $\mathcal{F}$. By Rice-Shapiro (direction: $\Rightarrow$), there is some finite $g \in \mathcal{F}$ such that $g \subseteq f$. However, no finite g can belong to $\mathcal{F}$, since g must be defined for all even inputs. Contradiction.

- $C \notin \mathcal{RE}$, by Rice-Shapiro. Indeed, otherwise, we let $\mathcal{F} = \{f \in \mathcal{R} | \forall x.\, f(2 \cdot x + 1) \text{ is not defined}\}$. Clearly, $C = \{n | \phi_n \in \mathcal{F}\}$. We have that the always undefined function $f_\emptyset$ belongs to $\mathcal{F}$. By Rice-Shapiro (direction: $\Leftarrow$), since $\mathsf{dom}(f_\emptyset) = \emptyset$ is finite, any computable super-function of $f_\emptyset$ belongs to $\mathcal{F}$. So, $g(n) = 5$ belongs to $\mathcal{F}$. Contradiction, since $g(1)$ is defined (as 5).

- D belongs to $\mathcal{R}$. To check whether a given x belongs to D, first we compute $y = \mathsf{proj1}(x)$ and $n = \mathsf{proj2}(x)$. If $y < 1000$, we return false. If $y = 1000$ then we need to check whether $\phi_n(n)$ halts in 1000 steps or fewer (note that k could be less than y in this case). This requires running the program for 1000 steps at most, so it can be computed. If $y > 1000$, we check whether $\phi_n(n)$ halts in y steps.

- E does not belong to $\mathcal{R}$. By contradiction, assume that V_E is a verifier for E. Then, we can compute the function

$$g(n) = \begin{cases} 1 & \text{if } V_E(\mathsf{pair}(y, n)) \text{ for some } 0 \leq y \leq 1000 \\ 0 & \textit{otherwise} \end{cases}$$

  We can see that g is the characteristic function of K. Indeed, if $n \notin \mathsf{K}$, then clearly all the 1000 checks performed by g fail, and $g(n) = 0$. Otherwise, if $n \in \mathsf{K}$, we consider two sub-cases.

  - If $\phi_n(n)$ halts in k steps, with $k \leq 1000$, then the verifier V_E must answer 1 on $\mathsf{pair}(\min(1000, k), n)$. Since $k \leq 1000$, that actually is $\mathsf{pair}(k, n)$. Again, since $k \leq 1000$, g checks for this case (that is we check the case $y = k$), so $g(n) = 1$.
  - If $\phi_n(n)$ halts in k steps, with $k > 1000$, then the verifier V_E must answer 1 on $\mathsf{pair}(\min(1000, k), n)$. Since $k > 1000$, that actually is $\mathsf{pair}(1000, n)$. Our g checks for this case, since y can be 1000, so $g(n) = 1$.

  E however belongs to $\mathcal{RE}$, since it is the range of the following partial recursive function:

$$h(x) = \begin{cases} \mathsf{pair}(\min(1000, \mathsf{proj2}(x)), \mathsf{proj1}(x)) \\ \qquad\qquad \text{if } \phi_{\mathsf{proj1}(x)}(\mathsf{proj1}(x)) \text{ halts in } \mathsf{proj2}(x) \text{ steps} \\ \text{undefined} \qquad\qquad \text{otherwise} \end{cases}$$

Exercise 7. *Prove or refute the following statements:*

- $f \notin \mathcal{R} \implies \mathsf{ran}(f) \notin \mathcal{RE}$
- $A \notin \mathcal{RE} \implies A \leq_m A \setminus \{3\}$

Answer.

- $f \notin \mathcal{R} \implies \mathsf{ran}(f) \notin \mathcal{RE}$ is false. Take $f = \chi_{\mathsf{K}}$, the characteristic function of the set K. We know that $f \notin \mathcal{R}$, but $\mathsf{ran}(f)$ is $\{0,1\}$ which is recursive.

- $A \notin \mathcal{RE} \implies A \leq_m A \setminus \{3\}$ is true. If $3 \notin A$, then the identity function is a m-reduction. Otherwise, assume $3 \in A$. We have $A \neq \{3\}$, otherwise A would be $\mathcal{RE}$. We can then pick an element $a \in A$, $a \neq 3$. We build the reduction function as follows:

$$f(x) = \begin{cases} x & \text{if } x \neq 3 \\ a & \text{otherwise} \end{cases}$$

  Obviously, $f \in \mathcal{R}$. If $x \in A$, with $x \neq 3$, then $f(x) = x \in A \setminus \{3\}$. If $x \in A$, with $x = 3$, then $f(x) = a \in A \setminus \{3\}$. If $x \notin A$, then $f(x) = x \notin A \setminus \{3\}$.

Exercise 8. *Let $\mathcal{R}^t$ be the set of total recursive functions. Prove or refute the following.*

$$\forall f \in \mathcal{R}^t.\, \exists n \in \mathbb{N}.\, \forall m \in \mathbb{N}, x \in \mathbb{N}. \quad \phi_{f(n,m)}(x) = \phi_n(m,x)$$

Answer. The statement is true. Given a total recursive f, we build n as follows. First, consider the following g, written in the usual notation:

$$g(y) = \#(\lambda m\, x.\, \phi_{f(y,m)}(x))$$

The function g is a recursive function, since we can define it using the universal program, general composition, and the usual $\mathbf{App}, \mathbf{Num}$ functions. function g is also total, by construction. From the second recursion theorem, for some n we have

$$\phi_{g(n)}(m,x) = \phi_n(m,x)$$

and, by definition of g, the left hand side is equal to $\phi_{f(n,m)}(x)$.