Computability Endterm Test — 2008-12-17

Exercise 1. *State whether $f \in \mathcal{R}$ where*

$$f(n) = \begin{cases} \phi_n(4) + 5 & \text{if } \phi_n(4) \text{ is defined} \\ 3 & \text{otherwise} \end{cases}$$

Justify your answer.

Answer. $f \notin \mathcal{R}$. By contradiction, assume $f \in \mathcal{R}$, so f is λ-defined by some F. Then $V_A = \lambda n.\, \mathbf{Eq}\ulcorner 3 \urcorner (Fn)$ is a verifier for the set

$$A = \{n \mid \phi_n(4) \text{ is not defined}\}$$

Indeed, when $n \in A$, then $f(n) = 3$ and $V_A \ulcorner n \urcorner = \mathbf{T}$. Otherwise, when $n \notin A$, then $f(n) \geq 5$, so $f(n) \neq 3$ and $V_A \ulcorner n \urcorner = \mathbf{F}$.

However this is a contradiction, since $A \notin \mathcal{R}$, which we can prove by Rice. Clearly, $\#\Omega \in A$, and $\#\mathbf{I} \notin A$. Also A is semantically closed: if $n \in A$ and $\phi_n = \phi_m$, then $\phi_n(4)$ is not defined, therefore $\phi_m(4)$ is not defined, so $m \in A$. □

Exercise 2. *State whether these sets are in $\mathcal{R}$, in $\mathcal{RE} \setminus \mathcal{R}$, or not in $\mathcal{RE}$. Justify your answers.*

- $A = \{n \mid \phi_n(5)$ *is defined and* $\phi_n(6)$ *is not defined*$\}$
- $B = \{n \mid \exists y.\ \phi_n(3 \cdot y)$ *is defined*$\}$

Answer (A). By Rice-Shapiro we show that $A \notin \mathcal{RE}$. First, note that $A = \{n \mid \phi_n \in \mathcal{F}\}$ where $\mathcal{F} = \{f \in \mathcal{R} \mid f(5) \text{ is defined but} f(6) \text{ is not}\}$. By contradiction, assume $A \in \mathcal{RE}$. Let g be the function such that $g(5) = 5$ and $g(x)$ is undefined for all $x \neq 5$. Clearly $\mathsf{dom}(g) = \{5\}$ is finite, and $g \subseteq id$, where id is the identity. By Rice-Shapiro, id belongs to $\mathcal{F}$, which is a contradiction because $id(6)$ is defined. □

Answer (B). $B \in \mathcal{RE}$, because

$$B = \{n \mid \exists k.\, \mathsf{pair}(n,k) \in B'\} \quad \text{where}$$
$$B' = \{\mathsf{pair}(n,k) \mid \text{running } \phi_n(3 \cdot \mathsf{proj1}(k)) \text{ halts in } \mathsf{proj2}(k) \text{ steps}\}$$

and B' is clearly recursive. Also, $B \notin \mathcal{R}$, which we prove by Rice. First, $\#\mathbf{I} \in B$ and $\#\Omega \notin B$. Then, B is semantically closed: when $n \in B$ and $\phi_n = \phi_m$, $\phi_n(3 \cdot y)$ is defined for some y, so $\phi_m(3 \cdot y)$ is defined for the same y, implying $m \in B$. □

Exercise 3. *Define* $C = \{2 \cdot n \mid n \in \mathbb{N}\}$. *Prove of refute the following statements.*

- $C \in \mathcal{RE}$
- $\forall D. \Big(D \subseteq C \implies \bar{\mathsf{K}} \leq_m D \cup \{2 \cdot n + 1 \mid n \notin \mathsf{K}\}\Big)$
- *If* $E = \{2 \cdot n \mid n \in \mathsf{K}\} \cup \{2 \cdot n + 1 \mid n \notin \mathsf{K}\}$, *then* $E \in \mathcal{RE}$.
- *If* E *is defined as above, then* $(\mathbb{N} \setminus E) \in \mathcal{RE}$

Answer (C). Clearly, $V_C = \textbf{Even}$ is a verifier, so $C \in \mathcal{R}$, therefore $C \in \mathcal{RE}$. □

Answer (D). The statement is true. Take $h(n) = 2 \cdot n + 1$, which is total recursive. Note that $D \cup \{2 \cdot n + 1 \mid n \notin \mathsf{K}\} = D \cup \{h(n) \mid n \in \bar{\mathsf{K}}\}$.

- if $n \in \bar{\mathsf{K}}$, then $h(n) \in \{h(n) \mid n \in \bar{\mathsf{K}}\}$ and so $h(n) \in D \cup \{h(n) \mid n \in \bar{\mathsf{K}}\}$.
- if $n \notin \bar{\mathsf{K}}$, then $h(n) \notin \{h(n) \mid n \in \bar{\mathsf{K}}\}$ since h is injective. Also, $h(n)$ is odd, so $h(n) \notin D$ because $D \subseteq C$. So, $h(n) \notin D \cup \{h(n) \mid n \in \bar{\mathsf{K}}\}$.

Therefore, $\bar{\mathsf{K}} \leq_m D \cup \{2 \cdot n + 1 \mid n \notin \mathsf{K}\}$. □

Answer (E). Since $\{2 \cdot n \mid n \in \mathsf{K}\} \subseteq C$, by (D), we get $\bar{\mathsf{K}} \leq_m E$, so $E \notin \mathcal{RE}$. □

Answer ($\bar{E}$). First, let *Even* be the set of even naturals, and *Odd* the set of the odd naturals. Since $f(i) = 2 \cdot i$ and $g(i) = 2 \cdot i + 1$ are injective, we have

$$\mathbb{N} \setminus \{2 \cdot n \mid n \in \mathsf{K}\} = \{2 \cdot n \mid n \notin \mathsf{K}\} \cup \mathit{Odd}$$
$$\mathbb{N} \setminus \{2 \cdot n + 1 \mid n \notin \mathsf{K}\} = \{2 \cdot n + 1 \mid n \in \mathsf{K}\} \cup \mathit{Even}$$

So,

$$\begin{aligned}\mathbb{N} \setminus E &= \mathbb{N} \setminus (\{2 \cdot n \mid n \in \mathsf{K}\} \cup \{2 \cdot n + 1 \mid n \notin \mathsf{K}\}) \\ &= (\mathbb{N} \setminus \{2 \cdot n \mid n \in \mathsf{K}\}) \cap (\mathbb{N} \setminus \{2 \cdot n + 1 \mid n \notin \mathsf{K}\}) \\ &= (\{2 \cdot n \mid n \notin \mathsf{K}\} \cup \mathit{Odd}) \cap (\{2 \cdot n + 1 \mid n \in \mathsf{K}\} \cup \mathit{Even}) \\ &= \{2 \cdot n \mid n \notin \mathsf{K}\} \cup \{2 \cdot n + 1 \mid n \in \mathsf{K}\}\end{aligned}$$

It is now easy to show that $\bar{K} \leq_m \bar{E}$ using the m-reduction $h(n) = 2 \cdot n$. Indeed, the proof is analogous to that of (D). This implies $\bar{E} \notin \mathcal{RE}$. □

Exercise 4. *Prove or refute the following statements.*

- $\forall A. \Big(A \in \mathcal{RE} \vee \bar{A} \in \mathcal{RE}\Big)$
- $\forall A. \Big(A \in \mathcal{RE} \implies \bar{A} \notin \mathcal{RE}\Big)$

Answer (1). The statement is false: taking $A = E$ from Ex. 3 suffices.

Alternatively, $A = \{n \mid \phi_n \text{ is total}\}$ also provides a counterexample, as can be shown by Rice-Shapiro. □

Answer (2). The statement is false. Take $A = \emptyset$: clearly $\emptyset \in \mathcal{RE}$ and $\mathbb{N} \in \mathcal{RE}$. □

Exercise 5. *Let f be a total recursive function such that*

$$\forall i, j \in \mathbb{N}. \left(i < j \implies f(i) < f(j)\right)$$

Is $\mathsf{dom}(f) \in \mathcal{R}$*? Is* $\mathsf{ran}(f) \in \mathcal{R}$*? Justify your answers.*

Answer. Trivially, $\mathsf{dom}(f) = \mathbb{N} \in \mathcal{R}$.

The function f is strictly increasing, and this implies that $f(i) \geq i$ for all i. Therefore, $n \in \mathsf{ran}(f)$ if and only if $n \in A = \{f(0), \ldots, f(n)\}$. Indeed,

- if $n \in A$, clearly $n \in \mathsf{ran}(f)$
- if $n \in \mathsf{ran}(f)$, then $n = f(i)$ for some i. By the fact above, $n = f(i) \geq i$, so $n \in A$.

Checking whether $n \in A$ can be effectively performed by computing the sequence $f(0), \ldots, f(n)$, which is possible because f is total, and comparing n with the elements of this sequence. So, $\mathsf{ran}(f) \in \mathcal{R}$. □

Note. Exercise 6 is **optional**. Solve it only if time allows.

Exercise 6. *Formally prove that if $g(-,-)$ is a function in $\mathcal{PR}$, then the function $f(x) = g(x, x)$ is in $\mathcal{PR}$. Starting from g and the initial functions, only, build f using the constructs in the definition of $\mathcal{PR}$. Make every use of composition or primitive recursion explicit.*

Answer. Consider the projection function $f_i(x_0, \ldots, x_n) = x_i$ when $i = n = 1$. That is just the identity function $id(x) = x$. By general composition applied to $g(-,-)$, $id(-)$, and $id(-)$ again, we have that

$$f(x) = g(id(x), id(x)) = g(x, x)$$

belongs to $\mathcal{PR}$. □