Galois Connections in Categorial Type Logic

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JOINT WORK WITH

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Contents

1. Introduction

- Categorial type logic provides a modular architecture to study **constants** and **variation** of grammatical composition:
  - **base logic** grammatical invariants, universals of form/meaning assembly;
  - **structural module** non-logical axioms (postulates), lexically anchored options for structural reasoning.
- Up till now, research on the constants of the base logic has focussed on (unary, binary, ...) **residuated** pairs of operators. E.g.
  - Value Raising: $A/C \vdash B/C$ if $A \vdash B$;
  - Lifting theorem: $A \vdash (B/A)\backslash B$.
- We extend the type-logical vocabulary with **Galois connected** operators and show how natural languages exploit the extra derivability patterns created by these connectives.

2. Residuated operators in categorial type logic

The connectives $\bullet B, /B$ and $A\bullet, A\backslash$ of NL in [Lambek 58, 61] form **residuated pairs** of operators, i.e. $\forall A, B, C \in \mathsf{TYPE}$,

$$[RES_2] \qquad A \vdash C/B \;\text{ iff }\; A \bullet B \vdash C \;\text{ iff }\; B \vdash A\backslash C$$

Similarly, the $\Diamond, \Box^{\downarrow}$ connectives introduced in [Moortgat 95] form a residuated pair, i.e. $\forall A, B \in \mathsf{TYPE}$,

$$[RES_1] \qquad \Diamond A \vdash B \;\text{ iff }\; A \vdash \Box^{\downarrow} B$$

3. Residuated and Galois connected functions

Consider two posets $\mathcal{A} = (A, \sqsubseteq_A)$ and $\mathcal{B} = (B, \sqsubseteq_B)$, and functions $f : A \to B$, $g : B \to A$. The pair (f, g) is said to be **residuated** iff $\forall a \in A, b \in B$

$$[RES_1] \qquad f(a) \sqsubseteq_B b \quad \text{iff} \quad a \sqsubseteq_A g(b)$$

The pair (f, g) is said to be **Galois connected** iff $\forall a \in A, b \in B$

$$[GC_1] \qquad b \sqsubseteq_B f(a) \quad \text{iff} \quad a \sqsubseteq_A g(b)$$

Remark Let $\mathcal{B}'$ be a poset s.t. $\mathcal{B}' = (B, \sqsubseteq'_B)$ where $x \sqsubseteq'_B y \stackrel{\text{def}}{=} y \sqsubseteq_B x$, and $h : B \to A$. If (f, h) is a residuated pair with respect to $\sqsubseteq_A$ and $\sqsubseteq'_B$, then it's Galois connected with respect to $\sqsubseteq_A$ and $\sqsubseteq_B$.

$$b \sqsubseteq_B f(a) \quad \text{iff} \quad f(a) \sqsubseteq'_B \quad \text{iff} \quad a \sqsubseteq_A h(b)$$

4. Models

Frames $F = \langle W, R_0^2, R_\diamond^2, R_\bullet^3 \rangle$

W: 'signs', resources, expressions

$R_\bullet^3$: 'Merge', grammatical composition

$R_\diamond^2$: 'feature checking', structural control

R_0^2: accessibility relation for the Galois connected operators

Models $\mathcal{M} = \langle F, V \rangle$

Valuation $V : \mathsf{TYPE} \mapsto \mathcal{P}(W)$: types as sets of expressions

5. Interpretation of the constants

$$
\begin{aligned}
V(\Diamond A) &= \{x \mid \exists y(R^2_\Diamond xy \;\&\; y \in V(A)\} \\
V(\Box^\downarrow A) &= \{x \mid \forall y(R^2_\Diamond yx \;\Rightarrow y \in V(A)\}
\end{aligned}
$$

$$
\begin{aligned}
V({}^{\mathbf{0}}A) &= \{x \mid \forall y(y \in V(A) \;\Rightarrow \neg R^2_0 yx\} \\
V(A^{\mathbf{0}}) &= \{x \mid \forall y(y \in V(A) \;\Rightarrow \neg R^2_0 xy\}
\end{aligned}
$$

$$
\begin{aligned}
V(A \bullet B) &= \{z \mid \exists x \exists y[R^3 zxy \;\&\; x \in V(A) \;\&\; y \in V(B)]\} \\
V(C/B) &= \{x \mid \forall y \forall z[(R^3 zxy \;\&\; y \in V(B)) \Rightarrow z \in V(C)]\} \\
V(A \backslash C) &= \{y \mid \forall x \forall z[(R^3 zxy \;\&\; x \in V(A)) \Rightarrow z \in V(C)]\}
\end{aligned}
$$

6. The base logic $\mathbf{NL}(\Diamond, \cdot^{0})$

Transitivity/Reflexivity of the derivability relation, plus

$$
\begin{array}{llll}
(\textsc{res-l}) & A \bullet B \vdash C & \text{iff} & A \vdash C/B \\
(\textsc{res-r}) & A \bullet B \vdash C & \text{iff} & B \vdash A\backslash C \\
\\
(\textsc{res-1}) & \Diamond A \vdash B & \text{iff} & A \vdash \Box^{\downarrow} B \\
\\
(\textsc{gal}) & A \vdash {}^{\mathbf{0}}B & \text{iff} & B \vdash A^{\mathbf{0}}
\end{array}
$$

Soundness/Completeness

$$A \vdash B \ \text{ is provable } \quad \text{iff} \quad \forall F, V,\ V(A) \subseteq V(B)$$

See [Areces, Bernardi & Moortgat 2001], also for Gentzen presentation, cut elimination and decidability.

7. Some useful derived properties

(Iso/Anti)tonicity $A \vdash B$ implies

$$\Diamond A \vdash \Diamond B \quad \text{and} \quad \Box^{\downarrow} A \vdash \Box^{\downarrow} B$$
$$^{\mathbf{0}}B \vdash {}^{\mathbf{0}}A \quad \text{and} \quad B^{\mathbf{0}} \vdash A^{\mathbf{0}}$$
$$A/C \vdash B/C \quad \text{and} \quad C\backslash A \vdash C\backslash B$$
$$C/B \vdash C/A \quad \text{and} \quad B\backslash C \vdash A\backslash C$$
$$A \bullet C \vdash B \bullet C \quad \text{and} \quad C \bullet A \vdash C \bullet B$$

Compositions

$$\Diamond\Box^{\downarrow} A \vdash A \qquad A \vdash \Box^{\downarrow}\Diamond A$$
$$A \vdash {}^{\mathbf{0}}(A^{\mathbf{0}}) \qquad A \vdash ({}^{\mathbf{0}}A)^{\mathbf{0}}$$
$$(A/B) \bullet B \vdash A \qquad A \vdash (A \bullet B)/B$$
$$B \bullet (B\backslash A) \vdash A \qquad A \vdash B\backslash(B \bullet A)$$

Closure Let $(\cdot)^*$ be ${}^{\mathbf{0}}(\cdot^{\mathbf{0}})$, $({}^{\mathbf{0}}\cdot)^{\mathbf{0}}$, $\Box^{\downarrow}\Diamond(\cdot)$, $X/(\cdot\backslash X)$, $(X/\cdot)\backslash X$. $\forall A \in$ TYPE we have

$$A \vdash A^*, \quad A^* \vdash B^* \text{ if } A \vdash B, \quad A^{**} \vdash A^*$$

8. Linguistic Applications

When looking at linguistic applications $\mathsf{NL}(\diamond, \cdot^{\mathbf{0}})$ offers:

- new (syntactic) derivability relations;
- new expressiveness on the semantic-syntactic interface;
- downward entailment relations.

We will show how

- the new patterns can be used to account for polarity items;
- the new relation on the syntactic-semantic interface sheds light on possible connections between dynamic Montague grammar and categorial type logic.

9. Polarity Items (I)

1. ***Any student** left.
2. **Some student** left.
3. John didn't see **any student**.
4. John didn't see **some student**.

Lexicon:

didn't:	$(np\backslash s)/(np\backslash({}^0s)^0)$
any N:	$q(np, ({}^0s)^0, ({}^0s)^0)$
some N:	$q(np, \Box^{\downarrow}\Diamond s, \Box^{\downarrow}\Diamond s)$

$$\dfrac{\dfrac{np \vdash np \quad s \vdash ({}^0s)^0}{np \circ np\backslash s \vdash ({}^0s)^0}(\backslash L) \qquad ({}^0s)^0 \not\vdash \Box^{\downarrow}\Diamond s}{\underbrace{q(np, ({}^0s)^0, ({}^0s)^0)}_{\text{Any student}} \circ \underbrace{np\backslash s}_{\text{left}} \vdash \Box^{\downarrow}\Diamond s}(qL)$$

$$q(np, s_1, s_2) \stackrel{\text{def}}{=} (np \rightarrow s_1) \rightarrow s_2$$

10. Polarity Items (II)

$\rightsquigarrow$ Wide Scope Negation ($\neg$GQ).

$$\dfrac{np \vdash np \qquad \dfrac{np \vdash np \quad s \vdash s_1}{np \circ np\backslash s \vdash s_1}\,(\backslash L)}{np \circ ((np\backslash s)/np \circ np) \vdash s_1}\,(/L)$$

$$\dfrac{np \circ ((np\backslash s)/np \circ np) \vdash s_1 \qquad s_2 \vdash ({}^0 s)^0}{np \circ ((np\backslash s)/np \circ q(np, s_1, s_2)) \vdash ({}^0 s)^0}\,(qL)$$

$$\dfrac{np \circ ((np\backslash s)/np \circ q(np, s_1, s_2)) \vdash ({}^0 s)^0}{(np\backslash s)/np \circ q(np, s_1, s_2) \vdash np\backslash({}^0 s)^0}\,(\backslash R)$$

$$\dfrac{np \vdash np \quad s \vdash \Box^{\downarrow}\Diamond s}{np \circ np\backslash s \vdash \Box^{\downarrow}\Diamond s}\,(\backslash L)$$

$$\dfrac{(np\backslash s)/np \circ q(np, s_1, s_2) \vdash np\backslash({}^0 s)^0 \qquad np \circ np\backslash s \vdash \Box^{\downarrow}\Diamond s}{\underbrace{np}_{\text{John}} \circ (\underbrace{(np\backslash s)/(np\backslash({}^0 s)^0)}_{\text{didn't}} \circ (\underbrace{(np\backslash s)/np}_{\text{see}} \circ \underbrace{q(np, s_1, s_2)}_{\text{GQ}})) \vdash \Box^{\downarrow}\Diamond s}\,(/L)$$

- GQ: **some student** $s_2 = \Box^{\downarrow}\Diamond s \qquad \Box^{\downarrow}\Diamond s \not\vdash ({}^0 s)^0$;
- GQ: **any student** $s_2 = ({}^0 s)^0 \qquad ({}^0 s)^0 \vdash ({}^0 s)^0$

11. Polarity Items (III)

$\rightsquigarrow$ Narrow Scope Negation (GQ$\neg$).

$$
\dfrac{
  \dfrac{
    np \vdash np
    \qquad
    \dfrac{
      \dfrac{s \vdash ({}^0s)^0 \qquad np \vdash np}{np\backslash s \vdash np\backslash({}^0s)^0}\,(\backslash R - L)
      \qquad
      \dfrac{np \vdash np \qquad s \vdash s_1}{np \circ np\backslash s \vdash s_1}\,(\backslash L)
    }{np \circ ((np\backslash s)/(np\backslash({}^0s)^0) \circ np\backslash s) \vdash s_1}\,(/L)
  }{np \circ ((np\backslash s)/(np\backslash({}^0s)^0) \circ ((np\backslash s)/np \circ np)) \vdash s_1}\,(/L)
  \qquad
  s_2 \vdash \Box^{\downarrow}\Diamond s
}{
  \underbrace{np}_{\text{John}} \circ (\underbrace{(np\backslash s)/(np\backslash({}^0s)^0)}_{\text{didn't}} \circ (\underbrace{(np\backslash s)/np}_{\text{see}} \circ \underbrace{q(np, s_1, s_2)}_{\text{GQ}})) \vdash \Box^{\downarrow}\Diamond s
}\,(qL)
$$

- GQ: **some student** $s_2 = \Box^{\downarrow}\Diamond s \qquad \Box^{\downarrow}\Diamond s \vdash \Box^{\downarrow}\Diamond s$;
- GQ: **any student** $s_2 = ({}^0s)^0 \qquad ({}^0s)^0 \not\vdash \Box^{\downarrow}\Diamond s$.

12. Typology of PIs

[Giannakidou 1997] extending the typology of PIs proposed in [van der Wouden 1994] considers them sensitive to non-veridicality. ($NV(p) \not\Rightarrow p$).

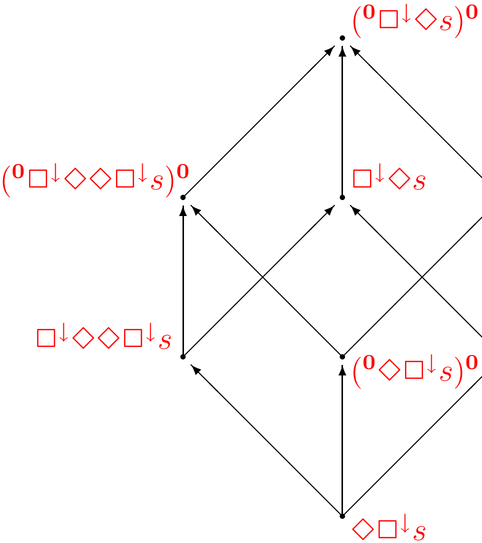
Thesis Episodic sentences (E) can be either veridical (V_{lic}) or non veridical (NV_{lic}). The latter contain the anti-veridical one (AV_{lic}) as subset. Negative polarity Items (NPIs) require AV_{lic}, whereas PIs NV_{lic}.

$$AV_{lic} : E/NPI \subseteq NV_{lic} : E/PI \quad \leadsto \quad PI \rightarrow NPI$$

$$\frac{\mathrm{AV}_{\mathrm{lic}} \in E/NPI \quad \mathrm{NPI} \in NPI}{\mathrm{AV}_{\mathrm{lic}} \circ \mathrm{NPI} \in E} \qquad \frac{\mathrm{NV}_{\mathrm{lic}} \in E/PI \quad \mathrm{PI} \in PI}{\mathrm{NV}_{\mathrm{lic}} \circ \mathrm{PI} \in E}$$

$$\frac{\dfrac{\mathrm{AV}_{\mathrm{lic}} \in E/NPI}{\mathrm{AV}_{\mathrm{lic}} \in E/PI} \quad \mathrm{PI} \in PI}{\mathrm{AV}_{\mathrm{lic}} \circ \mathrm{PI} \in E} \qquad \frac{\dfrac{\mathrm{NV}_{\mathrm{lic}} \in E/PI}{\mathrm{NV}_{\mathrm{lic}} \in E/NPI}[*] \quad \mathrm{NPI} \in NPI}{*\mathrm{NV}_{\mathrm{lic}} \circ \mathrm{NPI} \in E}$$

13. Options for cross-linguistic variation



14. Greek (I)

NPI: **ipe leksi**, PI: **kanenan**, FCI: **opjondhipote**

1. **Dhen** idha kanenan. — Neg > PI
   (tr. I didn't see anybody)
2. **Dhen** ipe leksi oli mera — Neg > NPI
   (tr. He didn't say a word all day)
3. *Dhen idha opjondhipote — *Neg > FCI
   (tr. I didn't see anybody)
4. Opjosdhipote fititis **bori** na lisi afto to provlima. — Modal > FCI
   (tr. Any student can solve this problem.)
5. **An** dhis tin Elena [puthena/optudhipote], . . . — Cond > PI/FCI
   (tr. If you see Elena anywhere, . . .)
6. **An** pis leksi tha se skotoso. — Cond > NPI
   (tr. If you say a word, I will kill you)

15. Greek (II)

The data presented above can be summarized as follows:

Greek	FCI	PI	NPI
Veridical	*	*	*
Negation	*	Yes	Yes
Modal verb	Yes	Yes	*
Conditional	Yes	Yes	Yes

Lexicon

PPI: $q(np, s_4, s_4)$, **kapjos**
PI: $q(np, s'_1, s'_1)$, **kanenan**
modal: $(((s'_4/np)\backslash s'_4)\backslash s_1)/(np\backslash s'_4)$, **bori**
cond.: $(s_1/s'_1)/s'_3$, **an**

NPI: $np\backslash s'_2$, **ipe leksi**
FCI: $q(np, s'_4, s'_4)$, **optudhipote**
neg.: $(np\backslash s_1)/(np\backslash s'_2)$, **dhen**

16. Italian (I)

NPI: **nessuno**, PI: **mai**, FCI: **chiunque**

1.	**Non** gioco mai (tr. I don't play ever)	Neg > PI
2.	**Non** ho visto nessuno (tr. I haven't seen anybody)	Neg > NPI
3.	***Non** ho visto chiunque (tr. I haven't seen anybody)	*Neg > FCI
4.	Chiunque **puó** risolvere questo problema (tr. Anybody can solve this problem)	Modal > FCI
5.	*Puoi giocare mai (tr. You can play ever)	*Modal > PI
6.	*Puoi prendere in prestito nessun libro (tr. You can borrow any book)	*Modal > NPI
7.	**Se** verrai mai a trovarmi, ... (tr. If you ever come to visit me, ...)	Cond > PI

17. Italian (II)

The data presented above can be summarized as follows:

Italian	FCI	PI	NPI
Veridical	*	*	*
Negation	*	Yes	Yes
Modal verb	Yes	*	*
Conditional	*	Yes	*

Lexicon

PPI: $q(np, s_4, s_4)$, **qualcuno**
PI: $(np\backslash s_1)\backslash(np\backslash s_1')$, **mai**
modal: $(((s_4''/np)\backslash s_4'')\backslash s_1)/(np\backslash s_4'')$, **puó**
cond: $(s_1/s_1')/s_4'$, **se**

NPI: $q(np, s_2', s_2')$, **nessuno**
FCI: $q(np, s_4'', s_4'')$, **chiunque**
neg.: $(np\backslash s_1)/(np\backslash s_2')$, **non**

18. The point up till now

These two examples show that the type hierarchy given by Galois and residuated unary operators

- helps carry out cross-linguistic analysis;
- predicts the existence of non veridical contexts which do not license polarity items, e.g. **possibly**, or non veridical contexts which license only some kind of PIs, but also PPIs, e.g. **puó** which license (only) FCIs, but also the PPI **qualcuno**;
- predicts the existence of some contexts shared by (negative) polarity items and positive one;
- sheds lights on new connections between dynamic Montague grammar and categorial type logic.

19. Connection with DMG

Non veridical (and therefore also anti-veridical) sentences do not allow anaphoric links. Veridical ones do.

1. This house **does not** have a bathtub.
   a) *It is/might be/possibly upstairs.
2. This house **might/could/should** have a bathtub.
   a) *It's green.
   b) It **might/could/should** be green.
3. This house **allegedly/possibly** has a bathtub.
   a) *It's green.
   b) It is **allegedly/possibly** green.

4

20. Conjecture

- If an expression is in the scope of $^{\mathbf{0}}(\cdot^{\mathbf{0}})$ it is closed;
- if it is in the scope of $\Box^{\downarrow}\Diamond\cdot$ anaphoric links are allowed.

Translating this into dynamic Montague grammar terms:

$$\begin{array}{llll} \Box^{\downarrow}\Diamond & \leadsto & \uparrow & \text{where } \uparrow\phi =_{def} \lambda p.(\phi \wedge {}^{\vee}p) \\ ^{\mathbf{0}}(\cdot^{\mathbf{0}}) & \leadsto & \downarrow & \text{where } \downarrow\psi =_{def} \psi({}^{\wedge}\mathsf{true}) \end{array}$$

21. Questions

- Can the connection with DMG help understanding the semantics of $({}^{\mathbf{0}}\cdot, \cdot^{\mathbf{0}})$?
- Is there any logic connection between Galois and non-veridicality vs. residuation and veridicality?
- So far we have being using only the composition of Galois operators. Hence, we have not use their downward monotonicity property. How could it be used in linguistic applications?

22. Conclusions

We have shown that

- the algebraic structure of $\mathsf{NL}(\Diamond)$ provides room for Galois connected operators in addition to the familiar residuated ones;
- residuated and Galois connected functions are closely related;
- extending $\mathsf{NL}(\Diamond)$ with unary Galois operators does not increase its complexity but *does* increase its expressiveness;
- the derivability patterns which characterize Galois connected and residuated operators give a proper typology of PIs and show new directions for linguistic investigation;
- on the other hand, the linguistic application considered opens the way to further logic research.