Yoneda (simplified)

$$\tau \simeq \forall \alpha.\, (\tau \to \alpha) \to \alpha$$

Proof: (informal)

$$f : \tau \to \forall \alpha.\, (\tau \to \alpha) \to \alpha$$

$$f = \lambda x:\tau.\, \Lambda \alpha.\, \lambda g : \tau \to \alpha.\, g\, x$$

$$f^{-1} : (\forall \alpha.\, (\tau \to \alpha) \to \alpha) \longrightarrow \tau$$

$$f^{-1} = \lambda h : (\forall \alpha.\, (\tau \to \alpha) \to \alpha).\, \underbrace{h[\tau]}_{(\tau \to \tau) \to \tau}\, (\lambda x:\tau.\, x)$$

Let's verify $f ; f^{-1} = id_\tau$

$\forall t : \tau$ we have

$$f^{-1}(f\, t) =_\beta f\, t^{x}\, [\tau]^{\alpha} (\lambda x : \tau.\, x)^{g}$$

$$=_\beta (\lambda x : \tau.\, x)\, t =_\beta t$$

We now need $f^{-1} ; f = id_{(\forall \alpha. \dots)}$

For this, we exploit <u>parametricity</u>

If $h : \forall \alpha.\, \underbrace{(\tau \to \alpha)}_{F\alpha} \longrightarrow \underbrace{\alpha}_{G\alpha}$

functorial

where:

$$\forall \ell : \alpha \longrightarrow \beta$$

$$F\ell : F\alpha \longrightarrow F\beta$$

$$(\tau \to \alpha) \to (\tau \to \beta)$$

$$w \longmapsto w ; \ell$$

$G\ell = \ell$ (G is the identity functor)

By naturality of h:

$$F\ell ; h_\beta = h_\alpha ; G\ell$$

i.e. $\forall w : \tau \to \alpha.$

$$h_\beta(F\ell(w)) = G\ell(h_\alpha(w))$$

$$h_\beta(w ; \ell) = \ell(h_\alpha(w))$$

$$h_\beta(w;\ell) = \ell(h_\alpha(w))$$

This holds $\forall \alpha.\ \forall w : \tau \to \alpha$

so, we choose $\alpha \equiv \tau$, $w = id_\tau : \tau \to \overset{\tau}{\underset{\alpha}{\shortparallel}}$

Hence:

$$h_\beta(id_\tau;\ell) = \ell(h_\tau(id_\tau))$$

$$h_\beta(\ell) = \ell(h_\tau(id_\tau)) \quad (*)$$

for all β, and for all $\ell : \tau \to \beta$

Let' verify $f^{-1}; f = id_{(\forall\alpha. \dots)}$

$\forall h : (\forall \alpha.\ (\tau \to \alpha) \to \alpha)$

$$f(f^{-1}(h)) =_\beta f(h_\tau(\lambda x : \tau. x))$$

$$=_\beta \Lambda\alpha.\ \lambda g : \tau \to \alpha.\ g(h_\tau(id_\tau))$$

$= ($by $*$, choosing $\beta \equiv \alpha$, $\ell \equiv g)$

$$\Lambda\alpha.\ \lambda g : \tau \to \alpha.\ h_\alpha(g)$$

$$=_\eta \Lambda\alpha.\ h_\alpha =_\eta h$$

Lemma (Yoneda, general)

In a parametric model, if $\sigma(\alpha)$ is functorial (w.r.t. α), for all τ:

$$\sigma(\tau) \simeq (\forall \alpha.\ (\tau \to \alpha) \to \sigma(\alpha))$$

(the simplified Yoneda uses $\sigma(\alpha) = \alpha$)

Proof (part)

$$f : \sigma(\tau) \to (\forall\alpha.\ (\tau \to \alpha) \to \sigma(\alpha))$$

$$f = \lambda x : \sigma(\tau).\ \Lambda\alpha.\ \lambda g : \tau \to \alpha.\ \underbrace{\sigma(g)}_{\sigma(\tau) \to \sigma(\alpha)}\, x$$

$$f^{-1} : (\forall\alpha.(\tau \to \alpha) \to \sigma(\alpha)) \longrightarrow \sigma(\tau)$$

$$f^{-1} = \lambda h : (\forall\alpha.(\tau \to \alpha) \to \sigma(\alpha)).\ \underbrace{h[\tau]}_{(\tau \to \tau) \to \sigma(\tau)}\,(\lambda x : \tau. x)$$

Ex: verify f, f^{-1} are inverses.

In category theory: the "real" lemma

Lemma (Yoneda, covariant)

Let $\mathcal{C}$ be a (locally small) category

Let $C \in |\mathcal{C}|$

and $F : \mathcal{C} \longrightarrow \mathrm{Set}$

We have $\mathcal{C}(C, -) : \mathcal{C} \longrightarrow \mathrm{Set}$

$$A \longmapsto \mathcal{C}(C, A)$$

$$(h : A \to B) \longmapsto (g \in \mathcal{C}(C, A) \mapsto (g ; h) \in \mathcal{C}(C, B))$$

then

$$F(C) \simeq_{\mathrm{Set}} [\mathcal{C}, \mathrm{Set}](\mathcal{C}(C, -), F)$$

$$\sigma(\tau) \simeq \forall \alpha \quad (\tau \to \alpha) \to F(\alpha)$$

naturally in C, F.