Parametricity

The interpretation of $[\![\forall \alpha . \tau]\!]^{\tau}$
is a family

$$\{ [\![\tau]\!]^{\tau}_{\theta[\alpha \mapsto A]} \}_{A \in |\mathcal{G}|}$$

Models of System F
are usually said to be:

- "ad hoc": the family is arbitrary
- "parametric": the family satisfies a fundamental property called <u>parametricity</u> / free theorem / abstraction theorem

We will see a simplified form of parametricity
(Possible project: the full form)

<u>Def</u> A type τ depending on $\alpha \in TV$ is functorial iff its interpretation $[\![\tau]\!]_{\theta[\alpha \mapsto A]}$ is a functor w.r.t. A
(covariant)

Examples: If $\alpha \notin fv(\sigma)$ then
$\tau(\alpha) = \sigma$, $\alpha + \sigma$, $\alpha \times \sigma$, $\sigma \to \alpha$,
$\alpha \times \alpha$, $\alpha + \alpha$, ...

Ex: $\tau(\alpha) = \sigma \to (\alpha + (\alpha \times (\sigma \to \alpha)))$

Non - example

$\tau(\alpha) = \alpha \to \sigma$ (is contravariant)

$= \alpha \to \alpha$

This notion extends to multiple variables, e.g.

$\tau(\alpha, \beta) = \alpha \times \beta \times \alpha$ is functorial wrt α, β

<u>Th</u> (Parametricity, simplified)

Let $\vdash t : \forall \vec{\alpha}. \tau(\vec{\alpha}) \to \sigma(\vec{\alpha})$

with τ, σ are functorial

then, in a parametric model,

$[\![t]\!]$ is a natural transformation

$$[\![t]\!] : [\![\tau]\!] \longrightarrow [\![\sigma]\!]$$

as functors of $\vec{\alpha}$

For instance, the PER model is parametric, hence the theorem applies to it

Example : let

$$f : \forall \alpha . \underset{\substack{\uparrow \\ I\alpha}}{\alpha} \to \underset{\substack{\uparrow \\ I\alpha}}{\alpha}$$

where I is the identity functor

By naturality (in a parametric model)

$\forall g : A \to B$

$$I g ; [\![f]\!]_B = [\![f]\!]_A ; I g$$

Let's write f for $[\![f]\!]$, with some abuse of notation.

$\forall A, B\ \forall g : A \to B, \quad g ; f_B = f_A ; g$

in particular

$\forall B \quad \forall x : 1 \to B \quad x ; f_B = f_1 ; x$

note that $f_1 : 1 \to 1 = !^1 = id_1$

hence

$\forall B \quad \forall x : 1 \to B \quad x ; f_B = x$

that is, informally :

$$f_B(x) = x \quad \forall B, \forall x : B$$

ie, f must be the (polymorphic) identity

This works because in $\lambda^\forall$ we can not write things like

$\Lambda \alpha .$ if $\alpha = \mathbb{N}$ then $\lambda x : \mathbb{N} . x + 4$
else $\lambda x : \alpha . \alpha$

which would violate parametricity

Example: $f : \forall \alpha.\ \alpha \longrightarrow \gamma \quad (\alpha \notin fv(\gamma))$

$$\underset{I\alpha}{\alpha} \longrightarrow \underset{\substack{K_\gamma \alpha \\ \text{constant functor}}}{\gamma}$$

$\forall g : A \to B.\quad Ig ; f_B = f_A ; K_\gamma g$

$$\Rightarrow \quad g ; f_B = f_A ; id_\gamma = f_A$$

In particular,

$\forall x : 1 \to B \quad x ; f_B = f_1$

Informally

$$f_B(x) = c \text{ for some } c : \gamma$$

Hence $f_B(x)$ does not depend on B or x : it must be a constant function!

Let $f : \forall \alpha.\ \alpha \times \gamma \longrightarrow \alpha \quad (\alpha \notin fv(\gamma))$

we can prove (informally) "$f(a, y) = a$"

i.e. $\forall a : 1 \to A,\ \forall y : 1 \to G = [\![\gamma]\!]$

$$\langle a, y \rangle ; f_A \overset{?}{=} a$$

$$\|$$

$$\langle id_1, y \rangle ; (a \times G) ; f_A$$

$$\|\ \text{nat.}$$

$$\underbrace{\langle id_1, y \rangle ; f_1}_{1 \longrightarrow 1 \times G \longrightarrow 1 \ \Rightarrow \text{ is } id_1} ; a$$

$$\|$$

$$id_1 ; a = a$$

Ex: $f : \forall \alpha, \beta.\ \alpha \times \beta \longrightarrow \beta \times \alpha$

then (informally) $f(a, b) = (b, a)$