Handling non-termination

term Ω_τ ($\approx$ while(true){})

type $\Gamma \vdash \Omega_\tau : \tau$

computational rule $\Omega_\tau \to_\beta \Omega_\tau$

"λ^Ω" (in addition to $\lambda^{\to}$, ...)

Consequences
1) Church-Rosser holds
2) Type uniqueness holds
3) Subject reduction holds
4) Weak normalization NO
5) Consistency of the associated logic (via C-H) NO

$$\vdash \Omega_0 : 0 \rightsquigarrow \vdash ff$$

We look for models of λ^Ω

Pos : category of posets
($\sqsubseteq$ refl., trans.
$x \sqsubseteq y \sqsubseteq x \Rightarrow x = y$ antisymm.)

objects : posets
morphisms : monotonic maps

In Pos

$1 = \{\bullet\}$ final

$A \times B$ pointwise ordered

$\langle a, b \rangle \sqsubseteq \langle a', b' \rangle \iff \begin{array}{l} a \sqsubseteq a' \\ b \sqsubseteq b' \end{array}$

A^B pointwise ordered

$\llcorner \{ f : B \to A \mid f \text{ monotonic} \}$

$f \sqsubseteq g \iff \forall b \in B.\ f(b) \sqsubseteq g(b)$

Ex: verify that Pos is a CCC

$\mathrm{Pos}(A \times B, C) \simeq \mathrm{Pos}(A, C^B)$

$\mathrm{Pos}_\perp$ posets having a $\perp$ element

Ex: $\mathrm{Pos}_\perp$ is a CCC

Th: $\mathrm{Pos}_\perp$ is a model of $\lambda^{\to, \times, \Omega}$

$[\![\Gamma \vdash \Omega_\tau : \tau]\!]^{\vdash} =$
$\lambda x \in [\![\Gamma]\!]^{\Gamma}.\ \perp_{[\![\tau]\!]^\tau}$

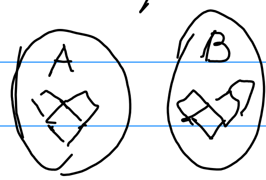
Example: interpret $\mathbb{B}$ (booleans) with the poset

$$\begin{array}{ccc} tt & & ff \\ & \perp & \end{array}$$

If $\vdash t : \tau \to \mathbb{B}$
and $[\![\vdash t\, \Omega_\tau : \mathbb{B}]\!] = (\bullet \mapsto tt)$
and $\vdash e : \tau$
then $[\![\vdash t\, e : \mathbb{B}]\!] = (\bullet \mapsto tt)$
by monotonicity !

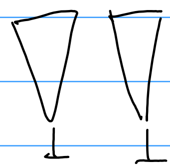
Note that Pos is also a bi-CCC

$A+B$ "side by side ordering"



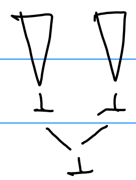
but $\mathrm{Pos}_\perp$ is not bi-

Naive sum

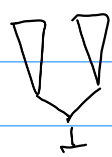


has no (unique) $\perp$

Lifted sum



Coalesced sum



neither is the coproduct

Handling (unconstrained) recursion

recursion ~ fixed point combinator Y

term Y_τ

type $\Gamma \vdash Y_\tau : (\tau \to \tau) \to \tau$

comp. rule $Y_\tau f \to_\beta f(Y_\tau f)$

(beyond $\lambda^{\to, \dots}$)

λ^Y (A.K.A. PCF)

As for λ^Ω, we lose weak normalization and consistency: $\vdash Y_0(\lambda x:0.\, x) : 0$

Example : Collatz

$$
\begin{aligned}
Collatz = Y_{\mathbb{N}\to\mathbb{N}} (& \lambda g : \mathbb{N}\to\mathbb{N}.\ \lambda n : \mathbb{N}. \\
& \text{if } n \leq 1 \text{ then } 1 \\
& \text{else if } n \text{ even then } 1 + g(n/2) \\
& \text{else } \qquad 1 + g(3n+1) \\
&)
\end{aligned}
$$

We look for models of λ^Y

Def An ω-CPO (ω-complete partial order) is a poset s.t. for any sequence

$$d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \quad (\omega\text{-chain})$$

we have a supremum $\bigsqcup_{n\in\mathbb{N}} d_n$

Def Let A, B be ω-CPOs.
We say that $f : A \to B$ is continuous iff $\forall d$ ω-chain in A

$$f\left(\bigsqcup^{A}_{n\in\mathbb{N}} d_n\right) = \bigsqcup^{B}_{n\in\mathbb{N}} f(d_n)$$

(iff f is continuous w.r.t. the Scott topology)

Lem $f : A \to B$ continuous $\Rightarrow$ f monotonic

Proof: Let $a \sqsubseteq a'$ in A.
Let $d : a \sqsubseteq a' \sqsubseteq a' \sqsubseteq a' \sqsubseteq a' \cdots$

then: $f(a) \underset{\text{sup}}{\sqsubseteq} \bigsqcup_n f(d_n) \underset{\text{cont}}{=} f\left(\bigsqcup_n d_n\right) = f(a')$

Lem ω-CPO is a bi-CCC

with

morphisms = continuous maps

$A^B = \{ f : B \to A \mid f \text{ continuous} \}$

(pointwise ordered)

(rest: as for Pos)

Ex: verify $\omega\text{-CPO}(A \times B, C) \simeq \omega\text{-CPO}(A, C^B)$

Def $\omega\text{-CPO}_\perp$ = ω-CPOs having a $\perp$ element

Th: $\omega\text{-CPO}_\perp$ is a CCC (non bi-)

Th: (Kleene's fixed point)

If $A \in |\omega\text{-CPO}_\perp|$, $f : A \to A$ cont.

then

$x = \bigsqcup_{n \in \mathbb{N}} f^n(\perp_A)$ is the least fixed point of f

(where $f^0(x) = x$

$f^{n+1}(x) = f(f^n(x))$)

Proof: $\{ f^n(\perp) \}_{n \in \mathbb{N}}$ is a ω-chain

$$\underbrace{\perp \sqsubseteq f(\perp)}_{\text{by } \perp} \sqsubseteq f^2(\perp) \sqsubseteq f^3(\perp) \ldots$$

f monot. f monot

...

$$f(x) \underset{\text{def } x}{=} f\Big(\bigsqcup_n f^n(\perp)\Big) \underset{f \text{ cont}}{=} \bigsqcup_n f(f^n(\perp))$$

$$= \bigsqcup_{n > 0} f^n(\perp) \overset{\perp \text{ does not matter}}{=} \bigsqcup_{n \geq 0} f^n(\perp) = x$$

x is least because, if y is a fixed point

$\perp \sqsubseteq y$ (by $\perp$)

$f(\perp) \sqsubseteq f(y) = y$ (by monot. f)

$f^2(\perp) \sqsubseteq f(y) = y$ (")

$\vdots$

$\forall n.\ f^n(\perp) \sqsubseteq y$

$\Rightarrow$ y is an upper bound of $\{f^n(\perp)\}_{n\in\mathbb{N}}$

$\Rightarrow$ $x = \bigsqcup_n f^n(\perp) \sqsubseteq y$

(sup = <u>least</u> upper bound)

<u>Th</u> ω-$CPO_\perp$ is a model of λ^Y

$$[\![\Gamma \vdash Y_\tau : (\tau \to \tau) \to \tau]\!]^\Gamma =$$

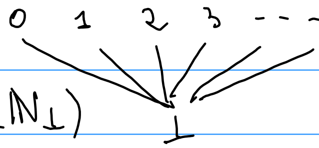
$$\lambda x \in [\![\Gamma]\!]^\Gamma .$$
$$\lambda f \in [\![\tau \to \tau]\!]^\tau . \underbrace{\bigsqcup_{n\in\mathbb{N}} f^n(\perp_{[\![\tau]\!]^\tau})}$$

(this is continuous wrt x and f)

Consequence:

- in λ^Ω we can only define monotonic functions
- in λ^Y we can only define continuous functions

Example: $[\![\mathbb{N}]\!]^\tau$



$(\mathbb{N}_\perp)$

$D = [\![\mathbb{N} \to \mathbb{N}]\!]^\tau$

$\perp_D = \lambda n \in \mathbb{N}_\perp . \perp_{\mathbb{N}_\perp}$

Let $[\![f]\!]: D \to D$ s.t. $Collatz = Y_{\mathbb{N}\to\mathbb{N}} f$

$\perp_D = \{ \_ \mapsto \perp \}$

$f(\perp_D) = \{ 0,1 \mapsto 1, \ \_ \mapsto \perp \}$

$f^2(\perp) = \{ 0,1 \mapsto 1, 2 \mapsto 2, \_ \mapsto \perp \}$

$f^3(\perp) = \{ \text{''}, 4 \mapsto 3, \_ \mapsto \perp \}$

$f^4(\perp) = \{ \text{''}, 8 \mapsto 4, \_ \mapsto \perp \}$

$f^5(\perp) = \{ \text{''}, 16 \mapsto 5, \_ \mapsto \perp \}$

$f^6(\perp) = \{ \text{''}, 32 \mapsto 6, 5 \mapsto 6, \_ \mapsto \perp \}$

$\circ \ \circ \ \circ \ \circ$

$h = [\![Collatz : \mathbb{N} \to \mathbb{N} \vdash Collatz(10) + Collatz(42)]\!]^\vdash$:

$[\![\mathbb{N} \to \mathbb{N}]\!]^\tau \longrightarrow [\![\mathbb{N}]\!]^\tau$

If $h(\bigsqcup_n f^n(\perp)) = K \neq \perp$

$\Rightarrow \exists \bar{n} \quad h(f^{\bar{n}}(\perp)) = K$ by contin.