Natural transformations form a category:

Let $\mathcal{C}, \mathcal{D}$ are categories, we can define a category $[\mathcal{C}, \mathcal{D}]$ or $(\mathcal{D}^{\mathcal{C}})$ where

$|[\mathcal{C}, \mathcal{D}]| = \{F : \mathcal{C} \to \mathcal{D} \mid F \text{ functor}\}$

$[\mathcal{C}, \mathcal{D}](F, G) = \{\eta : F \to G \mid \eta \text{ nat. tr.}\}$

Curry-Howard Isomorphism

Logical formulae:

- a atomic formula
- $p \to q$
- $p \wedge q$
- $p \vee q$
- tt, ff

<u>Def</u> $\neg p \equiv (p \to ff)$

Mapping from types to formulas

$$\ulcorner b \in BT \urcorner = b \quad (\text{atomic})$$
$$\ulcorner \tau \times \sigma \urcorner = \ulcorner \tau \urcorner \wedge \ulcorner \sigma \urcorner$$
$$\ulcorner \tau + \sigma \urcorner = \ulcorner \tau \urcorner \vee \ulcorner \sigma \urcorner$$
$$\ulcorner \tau \to \sigma \urcorner = \ulcorner \tau \urcorner \to \ulcorner \sigma \urcorner$$
$$\ulcorner 1 \urcorner = tt \qquad \ulcorner 0 \urcorner = ff$$

Lem: $\ulcorner - \urcorner$ is a bijection

We can also map contexts Γ to sets of formulae:

$$\ulcorner x_1 : \tau_1, \ldots, x_n : \tau_n \urcorner$$
$$= \ulcorner \tau_1 \urcorner, \ldots, \ulcorner \tau_n \urcorner$$

Finally, we can map judgments, by forgetting the λ-terms

$$\ulcorner \Gamma \vdash_\lambda t : \tau \urcorner = \ulcorner \Gamma \urcorner \vdash_{IPC} \ulcorner \tau \urcorner$$

IPC: intuitionistic propositional calculus

Natural Deduction Rules

$$\frac{}{\Gamma, p \vdash p}\ Ax$$

$$\frac{\Gamma, p \vdash q}{\Gamma \vdash p \to q}\ (\to I)$$

$$\frac{\Gamma \vdash p \to q \quad \Gamma \vdash p}{\Gamma \vdash q}\ (\to E)$$

$$\frac{\Gamma \vdash p \quad \Gamma \vdash q}{\Gamma \vdash p \wedge q}\ (\wedge I)$$

$$\frac{\Gamma \vdash p \wedge q}{\Gamma \vdash p}\ (\wedge E1) \qquad \frac{\Gamma \vdash p \wedge q}{\Gamma \vdash q}\ (\wedge E2)$$

$$\frac{\Gamma \vdash p}{\Gamma \vdash p \vee q}\ (\vee I1) \qquad \frac{\Gamma \vdash q}{\Gamma \vdash p \vee q}\ (\vee I2)$$

$$\frac{\Gamma \vdash p \vee q \quad \Gamma, p \vdash r \quad \Gamma, q \vdash r}{\Gamma \vdash r}\ (\vee E)$$

$$\frac{}{\Gamma \vdash tt}\ (tt\ I) \qquad \frac{\Gamma \vdash ff}{\Gamma \vdash p}\ (ff\ E)$$

<u>Th</u> (Curry - Howard)

$$(\exists t.\ \Gamma \vdash_\lambda t : \tau) \iff \ulcorner \Gamma \urcorner \vdash_{IPC} \ulcorner \tau \urcorner$$

<u>Proof</u> by the rules correspondence.

Curry Howard in more detail

types ~~~ formulae
terms ~~~ proofs
inhabited types ~~~ theorems

Ex: $((a \to b) \wedge a) \to b$

proof:
$\lambda x : (a \to b) \wedge a \,.\, \pi_1(x)\,(\pi_2(x))$

Ex: $p \to (q \vee (p \to q)) \to q$

proof:
$\lambda x : p .$
$\lambda y : q \vee (p \to q) .$
case y of
$in1(z : q) \to z$
$in2(w : p \to q) \to (w\, x)$

Ex: $(a \to b) \to (\neg b \to \neg a)$

proof:
$\lambda x : a \to b .$
$\lambda y : \neg b .$
$\lambda z : a .$
$y\,(x\, z)$

Lem If $b \in BT$
and $\Gamma \vdash_\lambda t : \tau$
then $\Gamma\{a/b\} \vdash_\lambda t\{a/b\} : \tau\{a/b\}$

Coroll: a atomic
$\Gamma \vdash p$ $\Rightarrow \Gamma\{q/a\} \vdash p\{q/a\}$

Lem $\nvdash_{\lambda} t : 0$

Proof: w.l.o.g. we can take t in normal form. The thesis follows by the normal form lemma.

Coroll: $\nvdash_{IPC} \bot$ (logic is consistent)