$$h : A \to C^B$$

$(\eta \to)$ $\qquad \lambda x . h x =_\eta h$

$$\Lambda(h \times id_B ; apply) = h$$

$$A \times B \to C^B \times B \to C$$

$$\Lambda(h \times id_B ; apply)$$

$=$ naturality Lem

$$h ; \Lambda(apply)$$

$=$ Lem

$$h ; id_{C^B} = h$$

<u>Th</u> In (bi-)CCC the following power laws

$$(A \times B)^C \simeq A^C \times B^C$$

$$A^{B \times C} \simeq (A^B)^C$$

$$A^{B+C} \simeq A^B \times A^C$$

$$A^1 \simeq A \qquad A^0 \simeq 1 \qquad 1^A \simeq 1$$

Proof: (for the first)
It suffices to work in the λ-calculus up to $\beta\eta$. Since CCCs respect $\beta\eta$ up to iso, we get the thesis

$$f : (C \to A \times B) \longrightarrow (C \to A) \times (C \to B)$$

(types)

$$f^{-1} : ((C \to A) \times (C \to B)) \to ((C \to A \times B))$$

$$f : (C \to A \times B) \longrightarrow ((C \to A) \times (C \to B))$$

$$f = \lambda x : C \to A \times B .$$
$$\langle \lambda y : C . \pi_1 (x y)$$
$$, \lambda y : C . \pi_2 (x y) \rangle$$

$$f^{-1} = \lambda x : ((C \to A) \times (C \to B)) .$$
$$\lambda y : C . \langle \pi_1(x) y , \pi_2(x) y \rangle$$

Check $f ; f^{-1} = id_{(C \to A \times B)}$

It suffices to check for $x \in Var$

$$f^{-1}(f \; x) = x \quad \forall x : C \to A \times B$$

[In fact $\Leftrightarrow \lambda x . f^{-1}(f x) = \lambda x . x$]

[$M =_{\beta\eta} N \Rightarrow \lambda x . M =_{\beta\eta} \lambda x . N$ (ξ)]

$$f^{-1}(f x) =_\beta \lambda y : C . \langle \pi_1 (f x) y , \pi_2 (f x) y \rangle$$
$$=_\beta \lambda y : C . \langle (\lambda y : C . \pi_1 (x y)) y$$
$$, (\lambda y : C . \pi_2 (x y)) y \rangle$$
$$=_\beta \lambda y : C . \langle \pi_1 (x y) , \pi_2 (x y) \rangle$$
$$=_{\eta \times} \lambda y : C . x y =_{\eta \to} x$$

Ex: $f(f^{-1} x) = x$

Naturality

Let $\{\eta_A\}_{A \in |\mathcal{C}|}$ be a family

$$\eta_A : FA \longrightarrow_{\mathcal{D}} GA$$

for two functors $F, G : \mathcal{C} \to \mathcal{D}$

Intuitively, η is natural iff it works in a "uniform" way for all A.

Es: $F, G : Set^2 \longrightarrow Set$

$$F(A, B) = A \times B$$
$$G(A, B) = B \times A$$

$$\eta_{A,B} : F(A,B) \to G(A,B)$$
$$\eta_{A,B} : A \times B \longrightarrow B \times A$$
$$\eta_{A,B}(a, b) = (b, a)$$

is natural

$$\begin{cases} \delta_{(\mathbb{Z},B)} : \mathbb{Z} \times B \to B \times \mathbb{Z} \\ \quad (a, b) \mapsto (b, a+1) \\ \delta_{(A \neq \mathbb{Z}, B)} : A \times B \to B \times A \\ \quad (a, b) \mapsto (b, a) \end{cases}$$

NOT natural

Def Let $F, G : \mathcal{C} \to \mathcal{D}$

$\{\eta_A\}_{A \in |\mathcal{C}|}$ is natural,

or a natural transformation

$$\eta : F \to G$$

iff

$$\forall f : A \to_{\mathcal{C}} B$$

$$\begin{array}{ccc} FA & \xrightarrow{Ff}_{\mathcal{D}} & FB \\ \downarrow \eta_A & & \downarrow \eta_B \\ GA & \xrightarrow{Gf}_{\mathcal{D}} & GB \end{array}$$

commutes

i.e. $Ff \,;\, \eta_B = \eta_A \,;\, Gf$

Es: $\eta_{A,B}(a, b) = (b, a)$ is NAT.

$$\forall f : (A, B) \longrightarrow_{Set^2} (A', B')$$

$$f = (f_1 : A \to A',\; f_2 : B \to B')$$

$$Ff \,;\, \eta_{A',B'} \overset{?}{=} \eta_{A,B} \,;\, Gf$$

$\Leftarrow \eta_{A',B'}(Ff(a, b)) = Gf(\eta_{AB}(a, b))$

$\Leftarrow \eta((f_1(a), f_2(b))) = Gf((b, a))$

$\Leftarrow (f_2(b), f_1(a)) = (f_2(b), f_1(a))$

✓

$\eta_A : A \times A \longrightarrow A \times A$

$F = G : Set \longrightarrow Set$

$FA = GA = A \times A$

$\eta(a_1, a_2) = (a_2, a_1)$

$\forall f : A \to_{Set} B \qquad Ff \,;\, \eta = \eta \,;\, Gf$

$= \eta \,;\, Ff$

Ex : $F : \mathbf{Set} \to \mathbf{Set}$

$A \longmapsto A^*$ set of finite sequences

$\langle a_1, \text{—}, a_n \rangle$ ($a_1, a_n \in A$)

$f : A \to B$

$Ff : A^* \to B^*$

$\langle a_1, \text{—} \rangle \longmapsto \langle f(a_1), \text{—} \rangle$

$\eta : F \longrightarrow F$

$\eta_A : A^* \longrightarrow A^*$

$\langle a_1, \text{—} \rangle \longmapsto \langle a_1, \text{—}, a_1, \text{—} \rangle$

Natural because $\forall f : A, B$

$\langle a_1, \text{—} \rangle \overset{Ff}{\longmapsto} \langle f(a_1), \text{—} \rangle$

$\overset{\eta_B}{\longmapsto} \langle f(a_1), \text{—}, f(a_1), \text{—} \rangle$

$\langle a_1, \text{—} \rangle \overset{\eta_A}{\longmapsto} \langle a_1, \text{—}, a_1, \text{—} \rangle$

$\overset{Ff}{\longmapsto} \langle f(a_1), \text{—}, f(a_1), \text{—} \rangle$

<u>Th</u> A cat $\mathcal{C}$ is a CCC iff

$\mathcal{C}$ has finite products $(1, \times)$

and

$\forall B, C \quad \exists$ "C^B" (not required to be the exponential)

$\exists \eta_A : \mathcal{C}(A \times B, C) \simeq_{\mathbf{Set}} \mathcal{C}(A, C^B)$

natural in A

(actually, it is also natural in A, B, C)

I.e. $\eta : F \longrightarrow G$

$F, G : \mathcal{C}^{op} \longrightarrow Set$

$F(A) = \mathcal{C}(A \times B, C)$

$G(A) = \mathcal{C}(A, C^B)$

$\forall f : A \longrightarrow_{\mathcal{C}} A'$

$Ff : \mathcal{C}(A' \times B, C) \rightarrow_{Set} \mathcal{C}(A \times B, C)$

$h \longmapsto f \times id_B ; h$

$Gf : \mathcal{C}(A', C^B) \longrightarrow_{Set} \mathcal{C}(A, C^B)$

$h \longmapsto f ; h$

Proof ($\Rightarrow$) We take C^B = exponential and

$\mathcal{C}(A \times B, C) \simeq \mathcal{C}(A, C^B)$

$\eta_A : \quad g \longmapsto \Lambda g$

$\eta_A^{-1} : \quad h \times id_B ; apply \longleftarrow h$

$\eta_A ; \eta_A^{-1} = id_{\mathcal{C}(A \times B, C)}$ for $[\beta]$ rule

$\eta_A^{-1} ; \eta_A = id_{\mathcal{C}(A, C^B)}$ for $[\eta]$ rule

η natural ?

$\forall f : A \to A' \quad Ff ; \eta_A = \eta_{A'} ; Gf$
(1) (2)

1) $h \xmapsto{Ff} f \times id_B ; h$
$\xmapsto{\eta} \Lambda(f \times id_B ; h)$

2) $h \xmapsto{\eta} \Lambda h$
$\xmapsto{Gf} f ; \Lambda h$

(1) = (2) by naturality lem

Proof We choose exponentials $= C^B$

Let $\eta_A : \mathcal{C}(A \times B, C) \simeq \mathcal{C}(A, C^B)$ nat

$\forall f : A \to A'$. $Ff ; \eta_A = \eta_{A'} ; Gf$

i.e. $\forall w : A' \times B \to C$

$$(Ff ; \eta_A)\, w = (\eta_{A'} ; Gf)\, w$$

i.e.

$$\eta_A(f \times id_B ; w) = f ; \eta_{A'}(w)$$

Since $id_{C^B} : C^B \longrightarrow C^B$

$$\eta^{-1}_{C^B}(id_{C^B}) : C^B \times B \longrightarrow C$$

We choose apply $= \circlearrowleft$

We need to prove that $\forall g : A \times B \to C$

$\exists !\ \Lambda g$ s.t. $[\beta]$ $\Lambda g \times id_B ; apply = g$

$$\Lambda g \times id_B ; apply = g$$

$\iff$ def apply

$$\Lambda g \times id_B ; \eta^{-1}_{C^B}(id_{C^B}) = g$$

$\iff$ η iso (bij.)

$$\eta_A(\Lambda g \times id_B ; \eta^{-1}_{C^B}(id_{C^B})) = \eta_A(g)$$

$\iff$ nat η ($f := \Lambda g$, $w = \eta^{-1}_{C^B}(id)$)

$$\Lambda g ; \eta_{C^B}(\eta^{-1}_{C^B}(id_{C^B})) = \eta_A(g)$$

$\iff$

$$\Lambda g = \eta_A(g)$$
