Properties of CCCs

Lem (naturality)

$\forall f : A \to B \;,\; g : B \times C \to D$

$$\Lambda(f \times id_C \,;\, g) = f \,;\, \Lambda g$$

Proof: by uniqueness of $(\Lambda -)$,
it suffices

$$(f ; \Lambda g) \times id_C \,;\, apply \overset{?}{=} f \times id_C \,;\, g$$

$\| $ functor $\times$

$$(f \times id_C) \,;\, \underbrace{(\Lambda g \times id_C) \,;\, apply}$$

$\|$ def CCC

$$(f \times id_C) \,;\, g$$

Lem: $apply : A^B \times B \to A$

$$\Lambda\, apply = id_{A^B} : A^B \to A^B$$

Proof: by uniqueness of $(\Lambda -)$
it suffices that

$$id_{A^B} \times id_B \,;\, apply = apply$$

$\|$ $\times$ functor

$$id_{(A^B \times B)} \,;\, apply = apply$$

Th Exponentials are unique, up to iso
More in detail, if we have
two exponentials for A^B
(C, Λ_C, ap_C)
(D, Λ_D, ap_D)
then $C \simeq D$

Proof $ap_C : C \times B \to A$ $\quad ap_D : D \times B \to A$
$\Lambda_D\, ap_C : C \to D$ $\quad \Lambda_C\, ap_D : D \to C$

We check $\Lambda_D\, ap_C ; \Lambda_C\, ap_D = id_C : C \to C$
(& the symm.)

$\Lambda_D\, ap_C ; \Lambda_C\, ap_D$
= by naturality
$\Lambda_C(\Lambda_D\, ap_C \times id_B ; ap_D)$
= by def CCC
$\Lambda_C(ap_C)$
= by lem
id_C

Def A CCC $\mathcal{C}$ is a bi-cartesian closed cat iff
- $0 \in |\mathcal{C}|$
- $A, B \in |\mathcal{C}| \Rightarrow A + B \in |\mathcal{C}|$

Th In a (bi-) CCC the β/η laws hold:

$(\to)$

β: $\Lambda g \times id \, ; apply = g$

η: $\Lambda(f \times id \, ; apply) = f$

$(\times)$

β: $\langle f, g\rangle ; \pi_1 = f$

$\langle f, g\rangle ; \pi_2 = g$

η: $\langle f;\pi_1 , f;\pi_2\rangle = f$

$(+)$

β: $in_1 ; [f, g] = f$

$in_2 ; [f, g] = g$

η: $[in_1 ; f , in_2 ; f] = f$

Hence a (bi) CCC is a model of $\lambda^{\to,\times}$ $({}^{,+})$

(& types)

<u>Th</u> Objects of a CCC form a commutative semiring up to ISO.

- $(1, \times)$ comm. monoid
- $(0, +)$ " " (if bi CCC)
- distributivity
  $A \times (B + C) \simeq (A \times B) + (A \times C)$
- annihilation
  $A \times 0 \simeq 0$

Proof of $0 \times A \simeq 0$

$f : 0 \times A \simeq 0 \quad = \pi_1$

$f^{-1} : 0 \simeq 0 \times A \quad = !_{0 \times A}$

$f^{-1} ; f : 0 \to 0 = !_0 = id_0$

$f ; f^{-1} : 0 \times A \simeq 0 \times A$

$=[\beta] \quad \underbrace{\Lambda(f ; f^{-1})}_{0 \to (0 \times A)^A} \times id_A ; apply$

$= \quad \underbrace{!_{(0 \times A)^0}}_{0 \to (0 \times A)^A} \times id_A ; apply$

$= \quad \Lambda(id_{0 \times A}) \times id_A ; apply$

$=[\beta] \quad id_{0 \times A}$