STLC

t typeable

$\Rightarrow$ $\exists$ normal form of t (normaliz.)

$\exists!$ " " " " (Church-Rosser)

& n.f. has the same (unique) type

Th: If $t \not\to_\beta$ normal form then:

$\emptyset \vdash t : \tau \to \sigma \Rightarrow t = \lambda x{:}\tau.\, e$ for some e

$\emptyset \vdash t : \tau \times \sigma \Rightarrow t = \langle e_1, e_2 \rangle$ " " e_1, e_2

$\emptyset \vdash t : 1 \Rightarrow t = \bullet$

$\emptyset \vdash t : \tau + \sigma \Rightarrow t = in1_\sigma(e)$ or $t = in2_\tau(e)$ for some e

$\emptyset \vdash t : 0 \Rightarrow$ false

Proof: by induction on $\vdash$ and case analysis

If $\vdash$ is an Intro rule, we get the thesis directly

If $\vdash$ is a Var rule, we get false since $\Gamma = \emptyset$

If $\vdash$ is an Elim rule, t is not a normal form. Some cases

($\vdash = \to E$)

$t = (e_1 e_2)$ by hp $t \not\to_\beta$

$\Rightarrow$ e_1, e_2 normal forms

$\emptyset \vdash e_1 : \tau \to \sigma$ $\emptyset \vdash e_2 : \tau$ for some τ, σ

by Induction Hp.

$e_1 = (\lambda x{:}\tau.\, e') \Rightarrow t \to_\beta e'\{e^2/x\}$

$(\times E1)$ $t = \pi_1(e)$

with $e \not\to_\beta$, $\emptyset \vdash e : \tau \times \sigma$ for some τ, σ

By I.H. $e = \langle e_1, e_2 \rangle$

$\Rightarrow t \longrightarrow_\beta e_1$

Other cases are similar.

Note that this does NOT hold for $\Gamma \neq \emptyset$!

$$x : \tau \times \sigma \vdash \underset{\langle e_1, e_2 \rangle}{\underset{\neq}{x}} \not\to_\beta \; : \tau \times \sigma$$

$$x : (\tau \times \sigma) \times \gamma \vdash \underset{\langle e_1, e_2 \rangle}{\underset{\neq}{\pi_1(x)}} \not\to_\beta \; : \tau \times \sigma$$

Ex: $\Gamma \vdash t : \tau$

$\Rightarrow var(\Gamma) \supseteq free(t)$

(by induction on t)

Denotational Semantics of STLC

There is a "trivial" set-theoretic model.

We fix $\theta : BT \longrightarrow Set$

Interpretation of types

$$\llbracket - \rrbracket^{\tau}_{\theta} : \mathrm{Type} \longrightarrow \mathrm{Set}$$

$$\llbracket \beta \rrbracket^{\tau}_{\theta} = \theta(\beta) \quad \text{if } \beta \in BT$$

$$\llbracket \tau \to \sigma \rrbracket^{\tau}_{\theta} = (\llbracket \tau \rrbracket^{\tau}_{\theta} \longrightarrow \llbracket \sigma \rrbracket^{\tau}_{\theta})$$

$$= \llbracket \sigma \rrbracket^{\tau\, \llbracket \tau \rrbracket^{\tau}_{\theta}}_{\theta}$$

$$\llbracket \tau \times \sigma \rrbracket^{\tau}_{\theta} = \llbracket \tau \rrbracket^{\tau}_{\theta} \times \llbracket \sigma \rrbracket^{\tau}_{\theta}$$

$$\llbracket 1 \rrbracket^{\tau}_{\theta} = 1 \quad (= \{\bullet\})$$

$$\llbracket \tau + \sigma \rrbracket^{\tau}_{\theta} = \llbracket \tau \rrbracket^{\tau}_{\theta} + \llbracket \sigma \rrbracket^{\tau}_{\theta}$$

$$\llbracket 0 \rrbracket^{\tau}_{\theta} = 0 \quad (= \emptyset)$$

We omit θ from now on.

Interpretation of terms

Version 1: "Winskel-style"

We use $\rho : \mathrm{Var} \to \bigcup_{\tau \in \mathrm{Type}} \llbracket \tau \rrbracket^{\tau}$

ρ "dynamic environment"

<u>Def</u>: environment update

$\rho[x \mapsto v] = \rho'$ where

$$\rho'(y) = \begin{cases} v & \text{if } y = x \\ \rho(y) & \text{o.w.} \end{cases}$$

$$[\![-]\!]^t_\rho \in \bigcup_{\tau \in Type} [\![\tau]\!]^\tau \quad \text{"denotation"}$$

$$[\![x]\!]^t_\rho = \rho(x)$$

$$[\![\lambda x : \tau . e]\!]^t_\rho = \lambda v \in [\![\tau]\!]^\tau . [\![e]\!]^t_{\rho[x \mapsto v]}$$

$$[\![(t\ e)]\!]^t_\rho = [\![t]\!]^t_\rho \left([\![e]\!]^t_\rho \right)$$

$$[\![\langle t, e \rangle]\!]^t_\rho = \left([\![t]\!]^t_\rho , [\![e]\!]^t_\rho \right)$$

$$[\![\pi_1(e)]\!]^t = \pi_1 \left([\![e]\!]_\rho \right)$$
$$[\![\pi_2(e)]\!]^t = \pi_2 \left([\![e]\!]_\rho \right)$$

$$[\![\bullet]\!]^t_\rho = \bullet \in 1$$

$$[\![in1(e)]\!]^t_\rho = in1 \left([\![e]\!]^t_\rho \right)$$
$$[\![in2(e)]\!]^t_\rho = in2 \left([\![e]\!]^t_\rho \right)$$

$$[\![case\ t\ of\ in1(x) \to e\ ;\ in2(y) \to f]\!]^t_\rho$$

$$= \begin{cases} \text{if } [\![t]\!]^t_\rho = in1(v) \\ \quad \text{then } [\![e]\!]^t_{\rho[x \mapsto v]} \\ \text{if } [\![t]\!]^t_\rho = in2(v) \\ \quad \text{then } [\![f]\!]^t_{\rho[y \mapsto v]} \end{cases}$$

$$[\![absurd_\tau(e)]\!]^t_\rho = !_{[\![\tau]\!]^\tau} \left([\![e]\!]^t_\rho \right)$$

<u>Th:</u> $t \to_{\beta\eta} t' \Rightarrow [\![t]\!]^t_\rho = [\![t']\!]^t_\rho$

$\vdash t : \tau \qquad \in [\![\tau]\!]^t_\rho$

Version 2 category-theoretic semantics

$\mathcal{C}$ a "suitable" category

$[\![-]\!]^{\tau} : \mathrm{Type} \longrightarrow \mathcal{C}$

as for Set $\quad \times =$ product
$\quad + =$ coproduct
$\quad 1 =$ final
$\quad 0 =$ initial

$\theta : BT \to \mathcal{C}$

Interpretation for static environments

$[\![-]\!]^{\Gamma} : \mathrm{StEnv} \longrightarrow \mathcal{C}$

$[\![\emptyset]\!]^{\Gamma} = 1$

$[\![\Gamma, x : \tau]\!]^{\tau} = [\![\Gamma]\!]^{\Gamma} \times [\![\tau]\!]^{\tau}$

We do NOT interpret terms, we interpret judgments

$[\![\Gamma \vdash e : \tau]\!]^{\vdash} : [\![\Gamma]\!]^{\Gamma} \xrightarrow[\mathcal{C}]{} [\![\tau]\!]^{\tau}$

We define $[\![-]\!]^{\vdash}$ for $\lambda^{\to, \times}$ only

$$[\![x_1 : \tau_1, \dots, x_n : \tau_n \vdash x_i : \tau_i]\!]^{\vdash}$$
$$= \pi_i^n = \pi_1 ; \pi_1 ; \dots ; \pi_2$$

$$((1 \times [\![\tau_1]\!]) \times [\![\tau_2]\!]) \times \dots \to [\![\tau_i]\!]$$

$$[\![\Gamma \vdash \langle t, e \rangle : \tau \times \sigma]\!]^{\vdash} =$$
$$\langle [\![\Gamma \vdash t : \tau]\!]^{\vdash}, [\![\Gamma \vdash e : \sigma]\!]^{\vdash} \rangle$$
$$: [\![\Gamma]\!]^{\Gamma} \to [\![\tau]\!]^{\tau} \times [\![\sigma]\!]^{\tau} = [\![\tau \times \sigma]\!]^{\tau}$$

$$[\![\Gamma \vdash \pi_1(e) : \tau]\!]^{\vdash} =$$
$$[\![\Gamma \vdash e : \tau \times \sigma]\!]^{\vdash} ; \pi_1$$
(similarly for π_2)

$$[\![\Gamma \vdash \bullet : 1]\!]^{\vdash} = !^{[\![\Gamma]\!]^{\Gamma}}$$

Note how we are defining $[\![-]\!]^{\vdash}$ by induction on the typing rules: e.g.

$$\frac{\overset{A}{\Gamma \vdash t : \tau} \qquad \overset{B}{\Gamma \vdash e : \sigma}}{\Gamma \underset{C}{\vdash} \langle t, e \rangle : \tau \times \sigma} \ (\times I)$$

we are assuming $[\![A]\!]^{\vdash}$, $[\![B]\!]^{\vdash}$ are already defined, and then we define $[\![C]\!]^{\vdash}$ as a function of $[\![A]\!]^{\vdash}$, $[\![B]\!]^{\vdash}$

We proceed in the same way for $\to$

$$\frac{\Gamma, x : \tau \vdash e : \sigma}{\Gamma \vdash \lambda x : \tau . e : \tau \to \sigma} \ (\to I)$$

Given

$[\![\Gamma, x : \tau \vdash e : \sigma]\!]$:

$[\![\Gamma]\!]^{\Gamma} \times [\![\tau]\!]^{\tau} \longrightarrow_{\mathcal{C}} [\![\sigma]\!]^{\tau}$

we need

$[\![\Gamma \vdash \lambda x : \tau . e : \tau \to \sigma]\!]$;

$[\![\Gamma]\!]^{\Gamma} \longrightarrow_{\mathcal{C}} [\![\sigma]\!]^{\tau\,[\![\tau]\!]^{\tau}}$

"exponential object"

We require $\mathcal{C}$ to be such that

$\forall A, B \in |\mathcal{C}| . \exists$ "B^A" $\in |\mathcal{C}|$

with $\forall ABC$

$\Lambda_{ABC} : \mathcal{C}(A \times B \to C) \to_{Set} \mathcal{C}(A, C^B)$

Λ_{ABC} bijection

Λ "Currying" (150)

$[\![\Gamma \vdash \lambda x : \tau . e : \tau \to \sigma]\!] =$

$\Lambda([\![\Gamma, x : \tau \vdash e : \sigma]\!])$

Given $[\![\Gamma \vdash t : \tau \to \sigma]\!]^{\vdash}$:

$[\![\Gamma]\!] \longrightarrow_{\mathcal{C}} [\![\sigma]\!]^{[\![\tau]\!]}$

and $[\![\Gamma \vdash e : \tau]\!]^{\vdash}$:

$[\![\Gamma]\!] \longrightarrow_{\mathcal{C}} [\![\tau]\!]$

we define $[\![\Gamma \vdash t\,e : \sigma]\!]$:

$[\![\Gamma]\!] \longrightarrow_{\mathcal{C}} [\![\sigma]\!]$

$= \langle [\![\Gamma \vdash t : \tau \to \sigma]\!], [\![\Gamma \vdash e : \sigma]\!] \rangle ; apply$

where $apply_{AB} : B^A \times A \longrightarrow_{\mathcal{C}} B$

This semantics is very general and can be used in any "cartesian closed category" (CCC)

<u>Def</u>: A CCC is a category $\mathcal{C}$ having

- $1 \in |\mathcal{C}|$
- $\forall A, B \in |\mathcal{C}|.\ A \times B \in |\mathcal{C}|$
- $\forall B, C \in |\mathcal{C}|.$ there is an

exponential object C^B equipped with
apply : $C^B \times B \longrightarrow C$ s.t.

$\forall g : A \times B \longrightarrow_{\mathcal{C}} C$
$\exists!\ \Lambda g : A \longrightarrow_{\mathcal{C}} C^B$ s.t.
$(\Lambda g \times id_B) ; apply = g$

$$\begin{tikzcd} A \times B \arrow[d, "g"'] \\ C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "\Lambda g"'] \\ C^B \end{tikzcd} \qquad \begin{tikzcd} A \times B \arrow[d, "\Lambda g \times id_B"'] \arrow[dd, bend left=60, "g"] \\ C^B \times B \arrow[d, "apply"'] \\ C \end{tikzcd}$$

$- \times - : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$
$\times(A, B) = A \times B$
$f : A \to A'$ $\qquad g : B \to B'$
$\times(f, g) : A \times B \to A' \times B'$
↳ $\langle \pi_1 ; f \ , \ \pi_2 ; g \rangle$