STLC

Simple Types

BT set of base types (may be $=\emptyset$)

$$\frac{\tau \quad \sigma}{\tau \to \sigma}$$ function type $\Big\}\ \lambda^{\to}$

$$\frac{\tau \quad \sigma}{\tau \times \sigma}$$ product type

$$\frac{}{1}$$ unit type / one / final $\Big\}\ \lambda^{\times}$

$$\frac{\tau \quad \sigma}{\tau + \sigma}$$ sum / coproduct type

$$\frac{}{0}$$ empty type / zero / void / null / initial $\Big\}\ \lambda^{+}$

Fragments $\lambda^{\to,\times}$, $\lambda^{\to,\times,+}$, ...

λ terms

$\lambda^{\to}$

$$\frac{}{x} \quad x \in \text{Vars}$$

$$\frac{t}{\lambda x{:}\tau \,.\, t} \quad \text{abstraction}$$

$$\frac{t \quad e}{(t\,e)} \quad \text{application}$$

$\lambda^{\times}$

$$\frac{t \quad e}{\langle t, e\rangle} \qquad \frac{t}{\pi_1(t)} \qquad \frac{t}{\pi_2(t)}$$

$$\frac{}{\bullet}$$

$$\lambda^{+} \left\{ \begin{array}{l} \dfrac{t}{in1_{\tau}(t)} \qquad \dfrac{t}{in2_{\tau}(t)} \qquad (l_1, l_2) \\[2ex] \dfrac{t \qquad e \qquad f}{\mathrm{case}\ t\ \mathrm{of}\ in1(x) \to e\ ;\ in2(y) \to f} \\[2ex] \dfrac{t}{absurd_{\tau}(t)} \quad \left(\begin{array}{c} \text{null / void / zero} \\ \text{- - - -} \end{array} \right) \end{array} \right.$$

Typing $\lambda^{\to}$

Γ static environment / context / basis

$\Gamma = x_1 : \tau_1, \ldots, x_n : \tau_n$

Var, Type — $x_1, \ldots, x_n$ distinct

In STLC, we consider Γ up-to permutations (Γ is a "set")

$\Gamma \vdash t : \tau$ typing assignment (judgment)

st. environ., term, type

"In the env. Γ, the term t has type τ"

$$\frac{}{\Gamma, x : \tau \vdash x : \tau}\ \text{Var}$$

$$\frac{\Gamma, x : \tau \vdash t : \sigma}{\Gamma \vdash (\lambda x : \tau .\, t) : \tau \to \sigma}\ (\to I)$$

$$\frac{\Gamma \vdash t : \tau \to \sigma \qquad \Gamma \vdash e : \tau}{\Gamma \vdash (t\, e) : \sigma}\ (\to E)$$

$\lambda^{\to}$: computational rules

$$(\lambda x:\tau.\,t)\,e \longrightarrow_\beta t\{e/x\}$$

$\approx$ β = " intro ; elim "

$$(\lambda x:\tau.\,t\,x) \longrightarrow_\eta t \quad \left(\begin{array}{c}\text{if}\\ x \notin free(t)\end{array}\right)$$

$\approx$ η = " elim ; intro "

$\underline{\lambda^{\times}}$

$$\frac{\Gamma \vdash t : \tau \qquad \Gamma \vdash e : \sigma}{\Gamma \vdash \langle t, e\rangle : \tau \times \sigma}\ (\times I)$$

$$\frac{\Gamma \vdash t : \tau\times\sigma}{\Gamma \vdash \pi_1(t) : \tau}\ (\times E1) \qquad \frac{\Gamma \vdash t : \tau\times\sigma}{\Gamma \vdash \pi_2(t) : \sigma}\ (\times E2)$$

$$\frac{}{\Gamma \vdash \bullet : 1}\ (1 I)$$

$$\pi_1(\langle t, e\rangle) \longrightarrow_\beta t$$
$$\pi_2(\langle t, e\rangle) \longrightarrow_\beta e$$

$$\langle \pi_1(t), \pi_2(t)\rangle \longrightarrow_\eta t$$

$\underline{\lambda^{+}}$

$$\frac{\Gamma \vdash t : \tau}{\Gamma \vdash in1_\sigma(t) : \tau+\sigma}\ (+I1)$$

$$\frac{\Gamma \vdash t : \sigma}{\Gamma \vdash in2_\tau(t) : \tau+\sigma}\ (+I2)$$

$$\frac{\Gamma \vdash t : \tau+\sigma \qquad \Gamma, x:\tau \vdash e : \gamma \qquad \Gamma, y:\sigma \vdash f : \gamma}{\Gamma \vdash \text{case } t \text{ of } in1(x) \to e;\ in2(y) \to f : \gamma}\ (+E)$$

$$\begin{array}{l} \text{case } \mathrm{in}_1(t) \text{ of} \\ \quad \mathrm{in}_1(x) \to e\,; \\ \quad \mathrm{in}_2(y) \to f \end{array} \xrightarrow[\beta]{} e\{t/x\}$$

$$\begin{array}{l} \text{case } \mathrm{in}_2(t) \text{ of} \\ \quad \mathrm{in}_1(x) \to e\,; \\ \quad \mathrm{in}_2(y) \to f \end{array} \xrightarrow[\beta]{} f\{t/y\}$$

$$\begin{array}{l} \text{case } t \text{ of} \\ \quad \mathrm{in}_1(x) \to \mathrm{in}_1(x) \\ \quad \mathrm{in}_2(y) \to \mathrm{in}_2(y) \end{array} \xrightarrow[\eta]{} t$$

$$\frac{\Gamma \vdash t : 0}{\Gamma \vdash \mathrm{absurd}_\tau(t) : \tau}\ (0E)$$

Ex: $\vdash \lambda x : \tau.\ \langle x, x \rangle : \tau \to \tau \times \tau$

$\vdash \lambda x : \tau \to \sigma.\ \lambda y : \tau.\ x\,y\ : (\tau \to \sigma) \to (\tau \to \sigma)$

$\nvdash \lambda x : \ .\ x\,x$ no type

$\vdash \lambda x : (\tau \to \tau) \to \sigma.\ x\,(\lambda y : \tau.\ y)$

$\quad : ((\tau \to \tau) \to \sigma) \to \sigma$

<u>Th</u>: $\lambda^{\to, \times, +}$ satisfies Church - Rosser

$$\begin{tikzcd} & t \arrow[dl, "*"', "\beta"] \arrow[dr, "*", "\beta"'] & \\ t' \arrow[dr, "*", "\beta"'] & & t'' \arrow[dl, "*"', "\beta"] \\ & e & \end{tikzcd}$$

$\exists$

Th: $\lambda^{\to,\times,+}$ is strongly normalizing

$$t \to_\beta \to_\beta \cdots\cdots \underset{\text{eventually}}{\to_\beta} e \not\to_\beta$$

Th: type uniqueness

$$\left.\begin{array}{l}\Gamma \vdash t : \tau \\ \Gamma \vdash t : \sigma\end{array}\right\} \Longrightarrow \tau = \sigma$$

Proof: by induction on $\vdash$ (or on t)

Th: Subject Reduction

$\Gamma \vdash t : \sigma$, $t \to_\beta t' \Rightarrow \Gamma \vdash t' : \sigma$

$(t \to_\eta t')$

Proof: By induction on $\vdash$

(& substitution lemma)

$$\left.\begin{array}{l}\Gamma, x : \tau \vdash t : \sigma \\ \Gamma \vdash e : \tau\end{array}\right. \Rightarrow \Gamma \vdash t\{e/x\} : \sigma$$