Collatz:

$Col = \lambda n.\ F\ n\ 0$

$F = \lambda n x.$ if $n \leq 1$
then x
else if even n
then $F(n/2)(x+1)$
else $F(3n+1)(x+1)$

recursive calls

Not a proper definition
we use F to define F ...

$G = \lambda f.\ \lambda n x.$
if $n \leq 1$ then x
else if even n then $f(n/2)(x+1)$
else $f(3n+1)(x+1)$

We want F s.t. $F = G\,F$

New goal: solving fixed point equations within the λ-calculus

A possible solution

$F = AA$ where $A = (\lambda w.\ G(ww))$

Check: $F = AA = (\lambda w.\ G(ww))\ A$
$= G(AA) = G\,F$

Fixed point combinators:

$$Y = \lambda f.\, (\lambda w.\, f(ww))(\lambda w.\, f(ww))$$

Church's paradoxical combinator

Prop: $\forall G.\ \ YG =_\beta G(YG)$

$$\Theta = (\lambda w f.\, f(wwf))(\lambda w f.\, f(wwf))$$

$$\forall G.\ \Theta G =_\beta G(\Theta G)$$

Ex: a recursive factorial

$$\begin{aligned} Fact = Y\,(\lambda f.\, \lambda n.\ & \text{if } n = 0 \text{ then } 1 \\ & \text{else } n \cdot (f(n-1))\ ) \end{aligned}$$

Def. a term t is weakly normalizing iff $t \to_\beta \cdots \to_\beta n \not\to_\beta$

n is said to be a normal form of t

Def. t is strongly normalizing iff $\neg(\underbrace{t \to_\beta \cdots \to_\beta \cdots}_{\infty \text{ steps}})$

Ex: $I = \lambda x.x \not\to_\beta$ s. norm.

$II \to_\beta I \not\to_\beta$ s. norm.

$\Omega \triangleq YI = (\lambda w.ww)(\lambda w.ww) \to_\beta \Omega$

NON w. norm.

$K = \lambda xy.x$

$$\begin{tikzcd} K\, I\, \Omega \arrow[d, "\beta"] \arrow[loop right, "\beta"] \\ (\lambda y. I)\,\Omega \arrow[d, "\beta"] \arrow[loop right, "\beta"] \\ I \arrow[d, "\beta", "/"'] \\ {} \end{tikzcd}$$

weakly but not strongly norm.

Th Church - Rosser "confluence"

"diamonds property"

If
$$\begin{tikzcd} & M \arrow[dl, "*"', "\beta"] \arrow[dr, "\beta", "*"'] & \\ N & & O \end{tikzcd}$$

then
$$\begin{tikzcd} N \arrow[dr, "\beta *"'] & & O \arrow[dl, "\beta *"] \\ & \exists P & \end{tikzcd}$$

Es:
$$\begin{tikzcd} & (\lambda x. x+x)(3\cdot 2) \arrow[dl, "*"'] \arrow[dr, "*"] & \\ 3\cdot 2 + 3\cdot 2 \arrow[dr, "*"'] & & (\lambda x. x+x)\, 6 \arrow[dl, "*"] \\ & 6+6 & \end{tikzcd}$$

Corollary: $\forall M$ has at most ONE normal form

Normalization strategy

Th: If M has a normal form N then $M \to_\beta^* N$ where each $\to_\beta$ step reduces the leftmost λ that is applied.

can be β reduced

$$(\lambda x.\ (\lambda y.\ \cdots)M)\ N$$

leftmost

$$(\lambda x.\ (\lambda y.\ \cdots)M)$$

leftmost

The Simply-Typed λ calculus
(STLC)

We want to avoid certain terms

if 5 then t else e // 5 not a boolean

S(S T) // T not a $\mathbb{N}$

((3 5) 7 // ((. .) not function

Allowed in the untyped λ