

Collatz:

$$G = \lambda n. F \ n \ 0$$

$$F = \lambda n x. \text{ if } n \leq 1$$

then x

else if even n

$$\text{then } F(n/2)(x+1)$$

$$\text{else } F(3n+1)(x+1)$$

recursive calls \rightarrow

Not a proper definition \curvearrowright

we use F to define $F \dots$

$$G = \lambda f. \lambda n x.$$

if $n \leq 1$ then x

else if even n then $f(n/2)(x+1)$

else $f(3n+1)(x+1)$

We want F s.t. $F = G F$

New goal: solving fixed point equations within the λ -calculus

A possible solution

$$F = A A \text{ where } A = (\lambda w. G(w w))$$

$$\begin{aligned} \text{Check: } F &= A A = (\lambda w. G(w w)) A \\ &= G(A A) = G F \end{aligned}$$

Fixed point combinators:

$$Y = \lambda f. (\lambda w. f(ww))(\lambda w. f(ww))$$

Church's paradoxical combinator

Prop: $\forall G. YG =_{\beta} G(YG)$

$$\Theta = (\lambda w f. f(wwf))(\lambda w f. f(wwf))$$

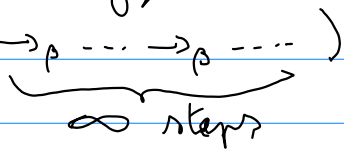
$$\forall G. \Theta G =_{\beta} G(\Theta G)$$


Ex: a recursive factorial

$$\text{Fact} = Y (\lambda f. \lambda n. \begin{cases} \text{if } n=0 \text{ then } 1 \\ \text{else } n \cdot (f(n-1)) \end{cases})$$

Def. a term t is weakly normalizing iff $t \rightarrow_{\beta} \dots \rightarrow_{\beta} n \not\rightarrow_{\beta}$

n is said to be a normal form of t

Def. t is strongly normalizing iff $\neg (t \rightarrow_{\beta} \dots \rightarrow_{\beta} \dots)$


Ex: $I = \lambda x. x \not\rightarrow_{\beta}$ s. norm.
 $II \rightarrow_{\beta} I \not\rightarrow_{\beta}$ s. norm.
 $\Omega \triangleq YI = (\lambda w. ww)(\lambda w. ww)$
 NON w. norm. 

$$K = \lambda x y. x$$

$$K \text{ I } \Omega \leftarrow \beta$$

$$\downarrow \beta$$

$$(\lambda y. \text{I}) \Omega \leftarrow \beta$$

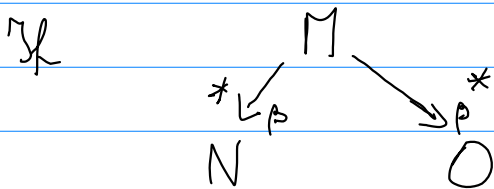
$$\downarrow \beta$$

$$\text{I}$$

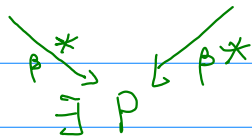
$$\downarrow \beta$$

weakly
but not
strongly
norm.

The Church - Rosser "confluence"
"diamonds property"



then



Ex: $(\lambda x. x+x)(3 \cdot 2)$

$$\begin{array}{ccc}
 * \swarrow & & \searrow^* \\
 3 \cdot 2 + 3 \cdot 2 & & (\lambda x. x+x) 6
 \end{array}$$

$$\begin{array}{ccc}
 * \rightarrow & & \leftarrow^* \\
 6 + 6 & &
 \end{array}$$

Corollary: λ M has at most ONE normal form

Normalization strategy

Th: \exists M has a normal form N

then $M \rightarrow_{\beta}^* N$ where each \rightarrow_{β} step reduces the leftmost λ that is applied.

can be β reduced

$(\lambda x. (\lambda y. \dots) M) N$

↑
leftmost

$(\lambda x. (\lambda y. \dots) M)$

↑
leftmost

The Simply-Typed λ calculus (STLC)

We want to avoid certain terms

if ω then t else e // ω not a boolean

$S(S T)$ // T not a \mathbb{N}

$(C \ \omega)$ // $(C \ \cdot)$ not function

↑
Allowed in the untyped λ