poset s.t.

$(A, \sqsubseteq)$ s.t. $\forall X \subseteq A.\ \exists \bigsqcup X$

↳ "complete lattice"

Ex: $(\mathcal{P}(B), \subseteq)$ $\bigsqcup = \bigcup$

Lem: In a CL $\forall X \subseteq A.\ \exists \bigsqcap X$

Ex: take $\bigsqcup lb(X)$ ....

Coroll: $(A, \sqsubseteq)$ CL $\Rightarrow (A, \sqsubseteq)^{op}$ CL.

Lem: In a CL,

$\bigsqcup X \sqsubseteq y \iff \forall x \in X.\ x \sqsubseteq y$

and the dual:

$x \sqsubseteq \bigsqcap Y \iff \forall y \in Y.\ x \sqsubseteq y$

Proof: (Ex.)

Th (Tarski) In a CL $(A, \sqsubseteq)$

$f : A \to A$ monotonic

$\Rightarrow \exists$ least (pre-) fixed point

Proof Take $x = \bigsqcap Y$ $Y = \{y \mid f(y) \sqsubseteq y\}$

Clearly, by def, $x \sqsubseteq$ any prefixed

We need to prove x is a prefixed

$f(x) \sqsubseteq x$

$\Leftarrow f(x) \sqsubseteq \bigsqcap Y$ [def x]

$\Leftarrow \forall y \in Y.\ f(x) \sqsubseteq y$ [Lem]

$\Leftarrow \forall y \in Y.\ f(x) \sqsubseteq f(y)$ [$f(y) \sqsubseteq y$]

$\Leftarrow \forall y \in Y.\ x \sqsubseteq y$ [f monotonic]

$\Leftarrow x \sqsubseteq \bigsqcap Y \Leftarrow$ true by def x [Lem]

Def: if $f : A \to A$ monotonic

$\mu f = \mu x. f(x) =$ least (pre-)fixed point

and, by duality

$\nu f = \nu x. f(x) =$ greatest (post-) fixed p.

Corollary:

Induction principle

$f(y) \sqsubseteq y \implies \mu f \sqsubseteq y$

Coinduction principle

$y \sqsubseteq f(y) \implies y \sqsubseteq \nu f$

$f(X) = \{0\} \cup \{x+2 \mid x \in X\}$

$\mu f = 2\mathbb{N}$ in $(\mathcal{P}(U), \subseteq)$

with U "large enough" (eg $\mathbb{R}$)

$f(Y) \subseteq Y \implies 2\mathbb{N} \subseteq Y$

$$\left\{\begin{array}{l} 0 \in Y \\ \forall x \in Y.\ x+2 \in Y \end{array}\right. \implies \forall y \in 2\mathbb{N}.\ y \in Y$$

---

trees $f(X) = \mathbb{N} \cup \{(l, r) \mid l, r \in X\}$

Induction:

$$\left\{\begin{array}{l} \forall n \in \mathbb{N}.\ p(n) \\ \forall l, r.\ p(l) \wedge p(r) \implies p((l, r)) \end{array}\right.$$

$\implies \forall t \in \mu f.\ p(t)$

Inference Rules

$$\frac{x_1 \; - \; - \; - \; x_n}{y}$$ ← premises (finitely many)

y ← consequence

R set of inference rules

$\hat{R}(X) = \{ y \mid \exists \frac{x_1 \cdots x_n}{y} \,.\, x_1, \dots, x_n \in X \}$

set of immediate consequences of X

<u>Prop</u>: $\hat{R}$ is monotonic

$X \subseteq Y \Rightarrow \hat{R}(X) \subseteq \hat{R}(Y)$

Coroll: $\exists \mu \hat{R}$
(on $(\mathcal{P}(U), \subseteq)$ CL where
U contains all the premises & consequences of all the rules in R)

Ex: $\left\{ \frac{}{0}, \frac{n}{n+1} \;\middle|\; n \in \mathbb{R} \right\} = R$

$\mu \hat{R} = \mathbb{N}$

$R = \left\{ \frac{}{0}, \frac{n}{n+2} \;\middle|\; \cdots \right\}$ $\mu \hat{R} = 2\mathbb{N}$

$R = \left\{ \frac{}{n}, \frac{l \quad r}{(l, r)} \;\middle|\; n \in \mathbb{N} \right\}$ $\mu \hat{R} = \text{Trees}$

$R = \left\{ \frac{}{n}, \frac{e_1 \quad e_2}{(e_1 + e_2)}, \frac{e_1 \quad e_2}{(e_1 - e_2)} \;\middle|\; n \in \mathbb{N} \right\}$

$\mu \hat{R} = $ arithmetic expressions

λ - calculus (untyped, or unityped)

$Var = \{x_0, x_1, \ldots\}$ countably infinite

λ-terms (λ-programs)

$$\frac{}{x}\,(x \in Var) \qquad \frac{t \quad e}{(t\,e)} \qquad \frac{t}{\lambda x . t}\,(x \in Var)$$

variable application abstraction

Ex. $\lambda x . x$ identity function

$\lambda x . y$ constant-y f.

$\lambda x . (\lambda y . x)$

$\lambda x . (\lambda y . y)$