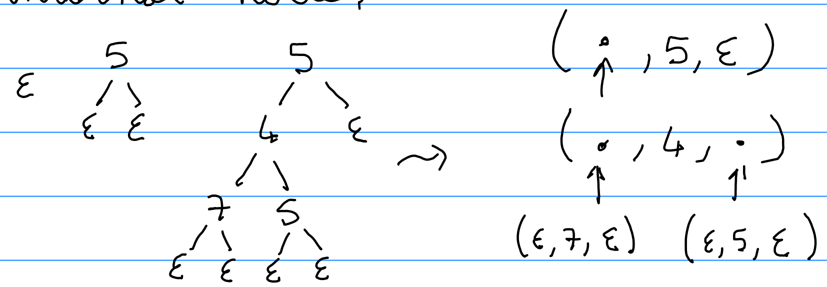
pre / post / fixed points

A "large enough" $(\mathcal{P}(A), \subseteq)$

$$f_4(X) = \{\varepsilon\} \cup \{(l, n, r) \mid n \in \mathbb{N} \wedge l, r \in X\}$$

binary trees with data in the internal nodes



least (pre-) fixed point

$(\mathcal{P}(\mathbb{N}^2), \subseteq) \circlearrowleft f_i$

$$f_5(R) = \{(0,0)\} \cup \{(x+1, y+2) \mid (x,y) \in R\} \cup \{(x, y+1) \mid (x,y) \in R\}$$

$S = \{(x,y) \mid y \geq 2x\}$

least (pre-) fixed point

$Q \in \mathcal{P}(\mathbb{N}^2)$ given

$f_6(R) = Q \cup \{(n,n) \mid n \in \mathbb{N}\}$
$\cup \{(x,z) \mid \exists y.\ (x,y),(y,z) \in R\}$

$S = \{(x,y) \mid \exists n \in \mathbb{N}, a_2, \dots, a_{n-1} \in \mathbb{N}.$
$(x,a_2),(a_2,a_3),\dots,(a_{n-1},y) \in Q\}$

S is the reflexive and transitive closure of Q

Ex: Q = successor $\quad S = (\leq)$

S is the least (pre-) fixed point of f_5.

Notation $\quad S = Q^*$ (Kleene's star)

$f_7(R) = Q \cup \{(n,n) \mid n \in \mathbb{N}\}$
$\cup \{(x,z) \mid \exists y.\ (x,y),(y,z) \in R\}$
$\cup \{(x,y) \mid (y,x) \in R\}$

least (pre-) fixed point =
smallest equivalence relation $\supseteq Q$

$f_8(R) = \{(0,1)\}$
$\cup \{(n+1, k(n+1)) \mid (n,k) \in R\}$

$S = \{(n, n!) \mid n \in \mathbb{N}\}$
least (pre-) fixed point

Note: $f_1, \dots, f_8$ are all monotonic

Under suitable orderings,
all monotonic functions have
least (pre-) fixed points (TODO)

Cat th

monot function ⇝ functor
prefixed points ⇝ algebra
postfixed " ⇝ coalgebra

$F : \mathcal{C} \to \mathcal{C}$ functor

F-algebra = (A, α) s.t. $A \in |\mathcal{C}|$

$\alpha : FA \to A$

" α is a witness for $f(a) \sqsubseteq a$ "

F-algebras form a category

$$|Alg_F| = \{ (A, \alpha) \mid A \in |\mathcal{C}| \wedge \alpha : FA \to_{\mathcal{C}} A \}$$

$$Alg_F((A, \alpha), (B, \beta)) = \{ f : A \to B \mid \alpha ; f = Ff ; \beta \}$$

$$\begin{array}{ccc} FA & \xrightarrow{\alpha} & A \\ \downarrow Ff & & \downarrow f \\ FB & \xrightarrow{\beta} & B \end{array}$$

In Set $\quad F : \mathrm{Set} \to \mathrm{Set}$

$$FX = X + 1$$

$h : A \to B$

$Fh : FA \to FB$

$\langle 1, a \rangle \mapsto \langle 1, h(a) \rangle$

$\langle 2, \cdot \rangle \mapsto \langle 2, \cdot \rangle$

$g : F\mathbb{N} \to \mathbb{N} \qquad (\mathbb{N}, g)$ F-alg

$g(\langle 1, n \rangle) = n+1$

$g(\langle 0, \cdot \rangle) = 0$

(g is isomorphism!)

$h : F\mathbb{N} \to \mathbb{N} \qquad$ F-alg

$h(\langle 1, n \rangle) = 2n+1$

$h(\langle 0, \cdot \rangle) = 0$

$f : (\mathbb{N}, g) \to (\mathbb{N}, h)$

$f(n) = 2^n - 1$

$$
\begin{tikzcd}
F\mathbb{N} \arrow[r, "g"] \arrow[d, "Ff"] & \mathbb{N} \arrow[d, "f"] \\
F\mathbb{N} \arrow[r, "h"] & \mathbb{N}
\end{tikzcd}
$$

$$Ff(\langle 1, n \rangle) = \langle 1, 2^n - 1 \rangle$$

$$Ff(\langle 2, \cdot \rangle) = \langle 2, \cdot \rangle$$

$$(f \circ g)(\langle 1, n \rangle) = f(n+1) = 2^{n+1} - 1$$

$$(f \circ g)(\langle 2, \cdot \rangle) = f(0) = 2^0 - 1 = 0$$

$$(h \circ Ff)(\langle 1, n \rangle) = h(\langle 1, 2^n - 1 \rangle)$$

$$= 2(2^n - 1) + 1$$

$$(h \circ Ff)(\langle 2, \cdot \rangle) = h(\langle 2, \cdot \rangle) = 0$$

$$\begin{tikzcd} F\mathbb{N} \arrow[r, "g"] \arrow[d, "Fl"] & \mathbb{N} \arrow[d, "\exists! l"] \\ FA \arrow[r, "\alpha"] & A \end{tikzcd}$$ (any F - alg)

ie, $(\mathbb{N}, g)$ is the <u>initial</u> F - alg

$l : (\mathbb{N}, g) \longrightarrow (A, \alpha)$

$l(0) = \alpha(\langle 2, \cdot \rangle)$

$l(n+1) = \alpha(\langle 1, l(n) \rangle)$

this definition is forced!

l is the unique morphism!

In Set, A set

$FX = A \times X + 1$

$\mathcal{L}$ = set of list (finite sequences) over A

$F\mathcal{L} \xrightarrow{g} \mathcal{L}$ F-alg.

$g(\langle 1, (a, l) \rangle) = a : l$

$g(\langle 2, \cdot \rangle) = \varepsilon$ (empty list)

$\mathcal{M} = \mathcal{L} \cup \{ a_1 : \cdots : a_m : x \mid x \in \mathbb{R} \wedge \forall i,\ a_i \in A \}$

$F\mathcal{M} \xrightarrow{h} \mathcal{M}$

$F(\langle 1, (a, m) \rangle) = a : a : m$

$F(\langle 2, \cdot \rangle) = \sqrt{2}$

$f : (\mathcal{L}, g) \longrightarrow (\mathcal{M}, h)$

$f(a_1 : \cdots : a_m : \varepsilon) = a_1 : a_1 : a_2 : a_2 : \cdots : a_m : a_m : \sqrt{2}$

$(\mathcal{L}, g)$ is the initial F-alg

Ex: $FX = A \times X \times X + 1$
has the algebra of binary trees (with data in the nodes) as initial algebra

Lem: $(A, \sqsubseteq)$ poset, $f : A \to A$ monotonic

a is the least prefixed point

$\Rightarrow$ a is the least fixed point

Consequence: to study the least fixed points, we can only search least prefixed points

Proof:

1)	$f(a) \sqsubseteq a$	[Hp]
2)	$f(f(a)) \sqsubseteq f(a)$	[monot.]
3)	$f(a)$ prefixed	[def]
4)	$a \sqsubseteq f(a)$	[a min]
5)	$f(a) = a$	[1, 4, antisymm]

$\Rightarrow$ a fixed point

(Min) b fix $\Rightarrow$ b prefix $\Rightarrow$ $a \sqsubseteq b$

Lem: $\mathcal{C}$ cat, $F : \mathcal{C} \to \mathcal{C}$ functor

(A, α) initial F-algebra

$\Rightarrow$ $\alpha : FA \simeq A$ iso

Proof:

$$\begin{tikzcd}
1) & FA \arrow[r, "\alpha"] \arrow[d, "F!_{FA}"'] \arrow[dr, phantom, "\simeq (1)"] & A \arrow[d, "4)\ !_{FA}"] & 3) \\
2) & F(FA) \arrow[r, "F\alpha"] \arrow[d, "F\alpha"'] \arrow[dr, phantom, "\simeq"] & FA \arrow[d, "\alpha"] & (FA, F\alpha)\ Alg. \\
5) & FA \arrow[r, "\alpha"] & A &
\end{tikzcd}$$

Let's check that $\alpha^{-1} = !_{FA}$

$!_{FA} ; \alpha = !_A = id_A$

$\alpha ; !_{FA} \underset{(1)}{=} F!_{FA} ; F\alpha = F(!_{FA} ; \alpha) = F(id_A) = id_{FA}$

F functor