Ex: $\times, +, 1, 0$

Let $\mathcal{C}$ be a cat with $0, 1, \times, +$

Prove that these are functors

$FX = A \quad (A \in |\mathcal{C}|)$

$FX = X$

$FX = GX \times HX \quad (G, H \text{ functors})$

$FX = GX + HX \quad$ "

$- \times - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

$- + - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

Lem: $(\times, 1)$ satisfies the monoidal laws up to ISO

$1 \times A \simeq A \times 1 \simeq A$

$(A \times B) \times C \simeq A \times (B \times C)$

Actually: the commutative monoidal laws

$f : A \times B \simeq B \times A$

$f = \langle \pi_2, \pi_1 \rangle = f^{-1}$

$f ; f^{-1} = f ; \langle \pi_2, \pi_1 \rangle$

$= \langle f ; \pi_2, f ; \pi_1 \rangle$

$= \langle \langle \pi_2, \pi_1 \rangle ; \pi_2, \dots \rangle$

$= \langle \pi_1, \dots \rangle$

$= \langle \pi_1, \langle \pi_2, \pi_1 \rangle ; \pi_1 \rangle$

$= \langle \pi_1, \pi_2 \rangle$

$= id_{A \times B}$

$f ; \langle g, h \rangle = \langle f ; g, f ; g \rangle$

$\langle g, h \rangle ; \pi_1 = g \qquad \dots = h$

$\langle \pi_1, \pi_2 \rangle = id_{A \times B}$

$1 \times A \simeq A$

$f : 1 \times A \to A$

$f = \pi_2 \qquad f^{-1} = \langle !^A , id_A \rangle$

$f ; f^{-1} = \pi_2 ; \langle !^A , id_A \rangle$

$= \langle \underbrace{\pi_2 ; !^A}_{1 \times A \to 1} , \pi_2 \rangle$

$= \langle !^{1 \times A} , \pi_2 \rangle$

$= \langle \pi_1 , \pi_2 \rangle = id_{1 \times A}$

$f^{-1} ; f = \dots$ (ex)

$f : (A \times B) \times C \simeq A \times (B \times C)$

$f = \langle \pi_1 ; \pi_1 , \langle \pi_1 ; \pi_2 , \pi_2 \rangle \rangle$

$f^{-1} = \langle \langle \pi_1 , \pi_2 ; \pi_1 \rangle , \pi_2 ; \pi_2 \rangle$

$(+, 0)$ satisfies the commutative monoidal laws (up to iso)

$(+, 0)$ in $\mathcal{C}$ $\equiv$ $(\times, 1)$ in $\mathcal{C}^{op}$

<u>Not</u> always true:

$A \times 0 \simeq 0$

True in Set but not in $\mathrm{Vect}_{\mathbb{K}}$

$0 = 1 = \{0\}$

$A \times 0 \simeq A \times 1 \simeq A$

$A \times (B + C) \simeq (A \times B) + (A \times C)$

True in Set, false in $\mathrm{Vect}_{\mathbb{K}}$

$+ = \times$

True in "distributive cat."

$\mathbb{R}^3 \not\simeq \mathbb{R}^4$

In general $in_1 : A \to A + B$
is not mono

$\equiv$ $\pi_1 : A \times B \to A$
is not epi

Esc in Set $\pi_1 : A \times \emptyset \to A$
is not surj (!)

Pre- / Post- / fixed points

$(A, \sqsubseteq)$ preorder (or poset)

$f : A \to A$

$f(x) = x$ x is a fixed point for f

$f(x) \sqsubseteq x$ prefixed

$f(x) \sqsupseteq x$ postfixed

Ex: 1) $(\mathcal{P}(\mathbb{R}), \subseteq)$ poset

$f_1(X) = \{0\} \cup \{x+2 \mid x \in X\}$

$\mathbb{R}, \mathbb{Q}, \mathbb{Z}, 2\mathbb{N}$ fixed points

least fix / prefix

$\mathbb{N}$ prefixed $f(\mathbb{N}) = \{0, 2, 3, \ldots\} = \mathbb{N} \setminus \{1\}$

$\{0\}$ postfixed $f(\{0\}) = \{0, 2\}$

2) $A = \{ x_1 : x_2 : \cdots : x_n : z$
$\mid x_1, - \in \mathbb{N} \wedge n \in \mathbb{N} \wedge$
$z \in \mathbb{R} \cup \{\varepsilon\} \}$

$(\mathcal{P}(A), \subseteq)$

$f_2(X) = \{\varepsilon\} \cup \{n : s \mid n \in \mathbb{N} \wedge s \in X\}$

$L = \{ x_1 : \cdots : x_n : \varepsilon \mid - - - \}$

least fixed point / prefixed point

$f(\emptyset) = \{\varepsilon\}$ postfixed

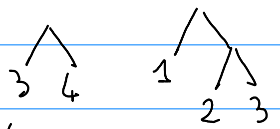
$f(\{\varepsilon\}) = \{\varepsilon, n : \varepsilon \mid n \in \mathbb{N}\}$ postfixed

$f(A) = A \setminus \mathbb{R}$ prefix

A "large enough" set $(\mathcal{P}(A), \subseteq)$

$f_3(X) = \mathbb{N} \cup \{(l, r) \mid l, r \in X\}$

$$T = \{ n, (n_1, n_2), (n_1, (n_2, n_3)), \\ ((n_1, n_2), n_3), \\ ((n_1, n_2), (n_3, n_4)), \ldots \}$$

binary trees 

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T least fixed/prefixed point