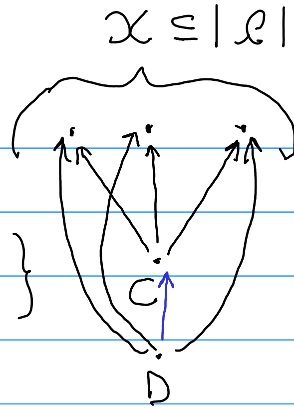
$\mathrm{Cone}_{\mathcal{X}}$ (simplified)
category:



$|\mathrm{Cone}_{\mathcal{X}}| = \{ (C, f_-) \mid \text{.. cone on } \mathcal{X} \}$

$\mathrm{Cone}_{\mathcal{X}}((C, f_-), (D, g_-))$
$= \{ h : D \xrightarrow[\mathcal{C}]{} C \mid \forall X \in \mathcal{X}.\ g_X = h ; f_X \}$

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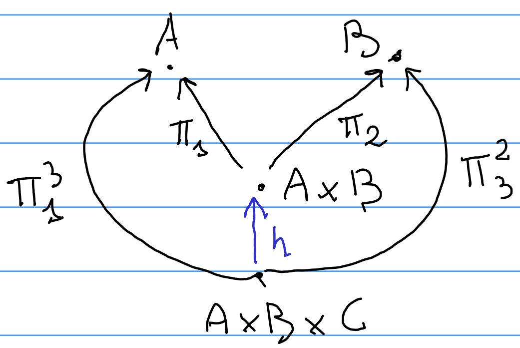
upper bound $ub(X)$ in $(A, \sqsubseteq)$, $X \subseteq A$
preorder

$\equiv$ lower bound in $(A, \sqsubseteq)^{op}$

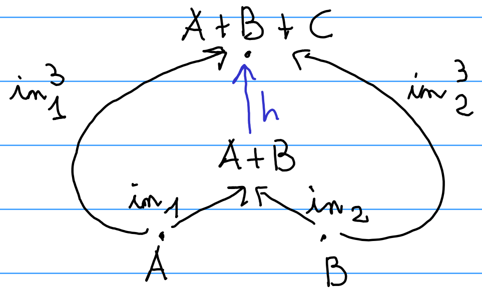
Co-cone in $\mathcal{C}$ $\equiv$ Cone in $\mathcal{C}^{op}$

Ex: in Set $\mathcal{X} = \{A, B\}$



$h(a, b, c) = (a, b)$

Let $A + B$ be the disjoint union
$A + B = (\{1\} \times A) \cup (\{2\} \times B)$



$in_1(a) = \langle 1, a \rangle$
$in_2(b) = \langle 2, b \rangle$

$h(\langle 1, a \rangle) = \langle 1, a \rangle$
$h(\langle 2, b \rangle) = \langle 2, b \rangle$
(no $\langle 3, c \rangle$)

$(A, \sqsubseteq)$ preorder

x is a "bottom" element $\equiv \forall a \in A.\ x \sqsubseteq a$

x "top" $\equiv$ x bottom in $(A, \sqsubseteq)^{op}$

$\perp$ bottom, $\top$ top

<u>Lem</u>: bottoms are unique, up to $\sim$

where $x \sim y \equiv x \sqsubseteq y \sqsubseteq x$

<u>Def</u>: a preorder $(A, \sqsubseteq)$ is a poset

iff $x \sim y \Rightarrow x = y$

(antisymmetry)

<u>Lem</u>: a preorder up to $\sim$ is a poset

poset $\rightsquigarrow$ cat up to iso

Cat th: $\bot \rightsquigarrow$ initial object

$X \in |\mathcal{C}|$ is weakly initial
$\equiv \forall C \in |\mathcal{C}| . \exists f : X \to C$
$X \in |\mathcal{C}|$ is initial
$\equiv \forall C \in |\mathcal{C}| . \exists ! f : X \to C$

final / terminal in $\mathcal{C}$ $\equiv$ initial in $\mathcal{C}^{op}$

Ex: In Set $\emptyset$ initial, $\forall a . \{a\}$ final
In Top " "
In $\mathrm{Vect}_{\mathbb{K}}$ $\{0\}$ initial and final
In Grp $\{e\}$ " " "

Notation: 0 initial
1 final
$!_A : 0 \to A$ $\quad$ $!^A : A \to 1$

Lem A, B initial $\Rightarrow A \simeq B$ (iso)

$$\begin{tikzcd} A \arrow[loop left, "!^A_A"] \arrow[r, shift left, "!^A_B"] & B \arrow[l, shift left, "!^B_A"] \end{tikzcd}$$

$!^A_B ; !^B_A = !^A_A = id_A$ (& symm.)

$\min^A(X) \equiv$ the $\perp$ in $(X, \sqsubseteq \cap X^2)$

$\max \equiv \min$ in $(A, \sqsubseteq)^{op}$

(min, max may not exist)

$\inf^A(X) = \sqcap^A X \equiv \max^A(lb^A(X))$

$\sup^A(X) = \sqcup^A X \quad = \sqcap X$ in $(A, \sqsubseteq)^{op}$

Cat th: inf $\rightsquigarrow$ final cone

sup $\rightsquigarrow$ initial cocone

In our simplified setting

inf $\rightsquigarrow$ product

sup $\rightsquigarrow$ coproduct (or sum)

In Set $A \times B$ is the product

$$\begin{tikzcd}
A & & B \\
& A \times B \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] & \\
& C \arrow[u, "\langle f, g \rangle"'] \arrow[uul, bend left, "f"] \arrow[uur, bend right, "g"'] &
\end{tikzcd}$$

$\langle f, g \rangle (c) = \langle f(c), g(c) \rangle$
is the unique morphism s.t.
$\langle f, g \rangle ; \pi_1 = f \qquad \langle f, g \rangle ; \pi_2 = g$

In Set $A+B$ is the coproduct

$$\begin{tikzcd} & C & \\ & A+B \arrow[u, "{[f,g]}"] & \\ A \arrow[ur, "in_1"] \arrow[uur, bend left=60, "f"] & & B \arrow[ul, "in_2"'] \arrow[uul, bend right=60, "g"'] \end{tikzcd}$$

$[f,g](\langle 1, a\rangle) = f(a)$

$[f,g](\langle 2, b\rangle) = g(b)$

is the unique morphism s.t.

$in_1 ; [f,g] = f \quad in_2 ; [f,g] = g$

In Top $\quad A \times B$ with the product topology
is the product (as in cat. th.)

In Grp $\quad A \times B$ (direct product)
is the product (as in cat. th.)

In $\mathrm{Vect}_{\mathbb{K}}$ ...

In $\mathrm{Vect}_{\mathbb{K}}$ $\quad A + B \simeq A \times B$

In Grp $\quad A + B \simeq A * B$
free product

$a\, b\, a'\, b' \cdots a''\, b''$

$\mathcal{X}$ infinite

in Set $\quad \prod \mathcal{X}$, $\coprod \mathcal{X}$ (disj. union)

in $\mathrm{Vect}_{\mathbb{K}}$ $\quad \prod \mathcal{X}$ is the product

$\vec{v} = \langle x_1, x_2, \ldots \rangle$

$\coprod \mathcal{X}$ is $\subseteq \prod \mathcal{X}$

$\vec{v}$ with finite support

$\mathcal{C}$ with $A \times B$

1) $\langle \pi_1, \pi_2 \rangle = id_{A \times B}$

is a cone morph.

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & B \\
& A \times B \arrow[ul, "\pi_1"] \arrow[u, "{\langle \pi_1, \pi_2 \rangle}"] \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

$\langle \pi_1, \pi_2 \rangle ; \pi_1 = \pi_1$

(& for π_2)

$$\begin{tikzcd}
A & A \times B & B \\
& C \arrow[ul, "f"] \arrow[u, "{\langle f, g \rangle}"] \arrow[ur, "g"'] &
\end{tikzcd}$$

but $\exists !$ cone mor. $A \times B \to A \times B$

by finality, which is $id_{A \times B}$

2) $f;\langle g, h\rangle = \langle f;g, f;h\rangle$

$$\begin{tikzcd}
A & & B \\
& A\times B \arrow[ul, "\pi_1"'] \arrow[ur, "\pi_2"] & \\
& C \arrow[u, "{\langle g;h\rangle}"'] \arrow[uul, "g"] \arrow[uur, "h"'] & \\
& D \arrow[u, "f"'] \arrow[uuul, bend left, "f;g"] \arrow[uuur, bend right, "f;h"'] \arrow[uu, bend left, "{\langle f;g, f;h\rangle}"] &
\end{tikzcd}$$

We verify that $f;\langle g, h\rangle$ is a cone morphism $D \to A\times B$, then by finality $f;\langle g,h\rangle = !^D = \langle f;g, f;h\rangle$

$f;\underbrace{\langle g,h\rangle;\pi_1}_{=g} \stackrel{?}{=} f;g$ (& for π_2)

$f, g : C \to A \times B$

3) $\begin{cases} f ; \pi_1 = g ; \pi_1 \\ f ; \pi_2 = g ; \pi_2 \end{cases} \Longrightarrow f = g$

$f = f ; id_{A \times B} \underset{1}{=} f ; \langle \pi_1, \pi_2 \rangle$

$\underset{2}{=} \langle f ; \pi_1 , f ; \pi_2 \rangle \underset{HP}{=} \langle g ; \pi_1 , g ; \pi_2 \rangle$

$\underset{2}{=} g ; \langle \pi_1, \pi_2 \rangle \underset{1}{=} g ; id_{A \times B} = g$