Set Theory: Sets vs Classes

$\{x \mid x \in A \wedge p(x)\}$ <u>set</u> (if A set)
$\{x \mid p(x)\}$ <u>class</u>
"proper class": class $\wedge \neg$ set

$Set = \{x \mid x \text{ set}\}$ proper class

$\hookrightarrow A = \{x \in Set \mid x \notin x\}$
$\hookrightarrow \ldots \to$ contradiction

$Grp = \{(A, \circ, e) \mid \ldots \text{ group}\}$
proper class

$f : Grp^2 \longrightarrow Grp$
$f(G, H) = G \times H$
$\hookrightarrow$ is NOT set: its dom(-) not a set
f is a "class-function"
$f = \{<<G, H>, G \times H> \mid G, H \in Grp\}$
is a proper class

$A = (Grp^2 \longrightarrow Grp)$
$f \in A$
A is NOT a set
If $(A, \circ, e)$ satisfies the group laws
--- it is <u>not</u> a group,
but a "class-group"

$Grp^A = \{G \mid G \text{ group}, |G| \leq A\}$
is a set (if A is a set)

$Set^1 = \{ A \mid card(A) \leq 1 \}$
is a proper class

If Set^1 set then $Set = \bigcup Set^1$
is a set

Preliminaries :
1) Domain Theory
2) Category Theory
3) The Untyped λ-calculus

Domain Th:

$(U, \sqsubseteq)$ is preorder iff
$\sqsubseteq \in \mathcal{P}(U^2)$
$\sqsubseteq$ reflexive & transitive

Ex: 1) $\mathcal{P}(\mathbb{N})$ ordered by card.
$\{1\} \sqsubseteq \{2,3\}$ $\quad \{1\} \sqsubseteq \{2\} \sqsubseteq \{1\}$
2) $(\mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}, \sqsubseteq)$ where
$A \sqsubseteq B$ iff $card(A) = card(B)$
$\wedge \ min(A) \leq min(B)$
3) $(\mathcal{P}(A), \subseteq)$

N.B. $x \sqsubseteq y \ \vee \ y \sqsubseteq x$
is NOT true, in general

Category Theory (as a generalization of preorders)

preorder "$x \leq y$" is true or false

cat. force to "justify" $x \leq y$

"g witnesses $x \leq y$"

A *category* $\mathcal{C}$ is given by

1) a class of *objects* $Ob(\mathcal{C})$ or $|\mathcal{C}|$ or $\mathcal{C}$ (abusing)

($\sim U$ in $(U, \leq)$)

2) $\forall A, B \in |\mathcal{C}|$, a class of *morphisms* or *arrows*

$f \in \mathcal{C}(A, B)$ or $f \in Hom_{\mathcal{C}}(A, B)$

or $f : A \xrightarrow[\mathcal{C}]{} B$

with the operations:

a) $\forall A \in |\mathcal{C}|$. $id_A \in \mathcal{C}(A, A)$

(id_A witnesses $A \leq A$)

"identity"

b) $\forall A, B, C \in |\mathcal{C}|$.

$f \in \mathcal{C}(A, B)$, $g \in \mathcal{C}(B, C)$

$\Longrightarrow$ $(f ; g) = (g \circ f) \in \mathcal{C}(A, C)$

($;$ witnesses transitivity of $\leq$)

"composition"

satisfying: $\forall f : A \xrightarrow[\mathcal{C}]{} B$. $id_A ; f = f ; id_B = f$

and associativity: $(f ; g) ; h = f ; (g ; h)$

Examples:

$(U, \subseteq)$ preorder $\Rightarrow \mathcal{U}$ cat

$|\mathcal{U}| = U$

$$\mathcal{U}(x,y) = \begin{cases} \{\cdot\} & \text{if } x \subseteq y \\ \emptyset & \text{o.w.} \end{cases}$$

$\mathcal{U}$ s.t. $\mathrm{card}(\mathcal{U}(x,y)) \leq 1$

$\Rightarrow \mathcal{U}$ is a preorder

- Set : sets & functions
- $\mathrm{Vect}_{\mathbb{K}}$: vector spaces on $\mathbb{K}$ & linear maps
- Grp : groups & homomorph.
- Top : topological spaces & continuous functions
- ⋮

"concrete categories"
sets with structure & structure-preserving maps

$(M, +, e)$ monoid $\Rightarrow \mathcal{M}$ cat

with $|\mathcal{M}| = \{\cdot\}$

$\mathcal{M}(\cdot,\cdot) = M$

$f;g = f+g \qquad id_{\cdot} = e$

$\mathcal{M}$ cat , $\mathrm{card}(|\mathcal{M}|) = 1$

$\Rightarrow \mathcal{M}$ is a monoid

Morphism $\neq$ function, in general

$(U, \sqsubseteq)$ preorder $\Rightarrow$

$(U, \sqsubseteq)^{op} = (U, \sqsupseteq)$ preorder

$\mathcal{C}$ cat $\Rightarrow$ $\mathcal{C}^{op}$ cat with

$|\mathcal{C}^{op}| = |\mathcal{C}|$

$\mathcal{C}^{op}(A, B) = \mathcal{C}(B, A)$

id_A in $\mathcal{C}^{op}$ = id_A in $\mathcal{C}$

$f ; g$ in $\mathcal{C}^{op}$ = $g ; f$ in $\mathcal{C}$

Special morphisms

- $f : A \to A$ "endo"
- f s.t. $g ; f = h ; f \Rightarrow g = h$ "mono"

  (in Set, mono $\Leftrightarrow$ injective)
- f s.t. $f ; g = f ; h \Rightarrow g = h$ "epi"

  (in Set, epi $\Leftrightarrow$ surjective)
- $f : A \to B$ s.t.

  $\exists g : B \to A$. $f ; g = id_A$

  $g ; f = id_B$

  f is "iso"

  (In Set iso $\Leftrightarrow$ bijection)

Esc : g is unique (f^{-1})

G group $\Rightarrow$ $\mathcal{G}$ cat. (as for monoids)

$\forall f : \bullet \xrightarrow[g]{} \bullet$. f iso

$\mathcal{C}$

$$\begin{tikzcd} \bullet \arrow[loop above, "id"] \arrow[r, "f"] & \bullet \arrow[loop above, "id"] \end{tikzcd}$$

$g ; f \Rightarrow g = id$

f mono $\wedge$ epi $\wedge$ $\neg$ iso

Preorders : monotonic functions

$f : (U, \sqsubseteq_U) \rightarrow (V, \sqsubseteq_V)$

monotonic iff

$\forall x, y \in U .\ x \sqsubseteq_U y \Rightarrow f(x) \sqsubseteq_V f(y)$

Pre category

$|Pre| = \{ (U, \sqsubseteq) \mid \text{preorders} \}$

$Pre(A, B) = \{ f : A \rightarrow B \mid f \text{ monotonic} \}$

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monotonic $\rightsquigarrow$ functors

$f : \underbrace{U \rightarrow V}_{\text{preorder}}$	$F : \underbrace{\mathcal{C} \rightarrow \mathcal{D}}_{\text{cat.}}$
$f(u) = v \in V$	$Fc = d \in \lvert\mathcal{D}\rvert$
$\forall u \in U$	$\forall c \in \lvert\mathcal{C}\rvert$
$u \sqsubseteq_U u'$	$g : c \rightarrow c'$
$\Downarrow$	$\Downarrow$
$f(u) \sqsubseteq_V f(u')$	$Fg : Fc \rightarrow Fc'$
	where
	$F id_c = id_{Fc}$
	$F(g;h) = Fg ; Fh$