Type Theory

Comp. Sci. $\leftrightarrow$ Programming Languages Theory

Logic

Geom.

Constructive Logics

Homotopy Type Theory

Set Th vs Type Th.

Frege : naive set th

1) p predicate
   $\hookrightarrow \exists A = \{x \mid p(x)\}$
2) $y \in \{x \mid p(x)\} \iff p(y)$
   (extensionality)

Russell's Paradox

$A = \{x \mid x \notin x\}$

(ext)

$\hookrightarrow \forall y.\ y \in A \iff y \notin y$

$A \in A \iff A \notin A$

$\hookrightarrow$ contradiction!

Russell's Type Theory
(informal ~)

↳ Constrain valid predicates

"Individuals"
Sets / Relations on
Sets / Relations on
⋮

(Relations had ramified types)

"If X is defined in terms of
all the elements of Y
then X can not be a part of Y"
Vicious - circle principle

↓

"$x \notin x$" is <u>meaningless</u>

$A = \{ n \in \mathbb{N} \mid \forall B \in \mathcal{P}(\mathbb{N}) .$
$\quad \ldots\ldots \}$

A —— $\mathcal{P}(\mathcal{P}(\mathbb{N}))$
$\mathcal{P}(\mathbb{N})$
A $\not\to$ $\mathbb{N}$

A can not be at this level

Principia Mathematica ✻12
Axiom of Reducibility

Ax. Reduc:

$\forall p$ (of x at type t).
  $\exists p'$ (" " " " ").
    $p' \Leftrightarrow p \wedge p'$ has type "$t+1$"

$\approx \exists A = \{n \mid p(n)\}$

Modern Set Theory

ZF(C): Zermelo & Fraenkel

Informally:

$\exists$ sets $\emptyset, \{x\}, \{x, y\}$
$\exists \quad A \cup B, A \cap B, A \times B, \mathcal{P}(A)$
$\quad \bigcup_i A_i, \bigcap_i A_i, (A \to B), \ldots$

$\forall p, A$ set. $\exists B = \{x \in A \mid p(x)\}$

Axiom of Separation
↳ No more Russell's Paradox

$\mathbb{N}, \mathbb{N} \to \mathbb{N}$
↳ $\mathbb{N} \cup (\mathbb{N} \to \mathbb{N})$
  ↳ $\{5, 42$, id, sqr, $\ldots\}$