Type Theory – 2017-09-08

Exercise 1. *Provide the typing rules for the Simply Typed Lambda Calculus related to* $\cdot \to \cdot$ *and* $\cdot \times \cdot$ *types.*

Exercise 2. *Replace the placeholders below so to make the following typing judgments correct, in the Calculus of Constructions.*

$$\begin{array}{ll} 1) & \alpha : *,\ \beta : *,\ P : \beta \to *, \\ & h : \prod_{f:\alpha\to\beta} \prod_{a:\alpha} \prod_{b:\beta} ((P(fa) \to Pb) \to Pb) \\ & \vdash \boxed{?} : \alpha \to \prod_{b:\beta} Pb \\ 2) & \alpha : *,\ x : \boxed{?}\ \vdash x \boxed{?} (x \boxed{?}) : \alpha \end{array}$$

Exercise 3.

1. *In a cartesian closed category* $\mathcal{C}$*, let* A *be an object and take* $F(X) = X^A$*. Extend* F *so to form a functor* $F : \mathcal{C} \to \mathcal{C}$*. You can omit the verification of the functor laws.*

2. *Then, consider System F extended with coproducts. Let* τ, σ *two types with no free occurrence of* α*, and let:*

$$f : \forall\alpha.\ (\tau + \alpha) \to (\sigma \to \alpha)$$

Provide the naturality law associated to the interpretation of f *in a parametric model. Simplify this law so that it only involves the categorical constructs* $[-,-], in_1, in_2, (;), \Lambda, \mathsf{apply}$ *and variables.*

Exercise 4. *Let* $\mathcal{C}$ *be a cartesian closed category with an object* A*. Prove that*

$$\langle !^A; \Lambda(\pi_2; \mathsf{apply}^{A^A}), \langle !^A; \Lambda\pi_2, id_A \rangle\rangle; \mathsf{apply}^{A^{(A^A \times A)}} = id_A$$

Exercise 5. *Let* $\mathcal{C}$ *be a category such that for every endofunctor* $G : \mathcal{C} \to \mathcal{C}$ *there exists an initial* G*-algebra* $(X \in |\mathcal{C}|, f : GX \to X)$*.*

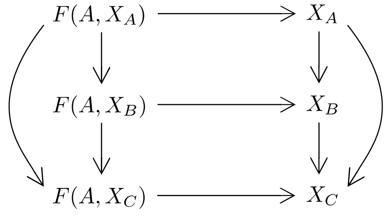
Let F *be a functor* $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$*. For every object* $C \in |\mathcal{C}|$*, we have that* $G = F(C, -)$ *is an endofunctor* $\mathcal{C} \to \mathcal{C}$*, hence admitting an initial algebra* $(X_C, f_C : F(C, X_C) \to X_C)$*.*

Prove that $X_- : \mathcal{C} \to \mathcal{C}$ *is a functor, following these steps:*

1. *Obviously* X_- *maps objects to objects. Define the other half of the functor, showing how to map any* $g : A \to B$ *to a morphism* $X_g : X_A \to X_B$*. This can be done by crafting a* $F(A, -)$*-algebra of the form* $(X_B, ??)$*, and exploiting initiality. This also forms a commutative square, providing a useful commuting property involving* X_g*. Simplify this equation so that it contains a single occurrence of* F*.*

2. *Prove that, for any* $A \in |\mathcal{C}|$*, we have* $X_{id_A} = id_{X_A}$*, exploiting initiality.*

3. *Prove that, for any* $A, B, C \in |\mathcal{C}|$*,* $g : A \to B$*,* $h : B \to C$ *we have* $X_{g;h} = X_g; X_h$*. For this, reason on the following diagram:*



   (a) *Find the morphisms above. Choose the three rightmost morphisms so that, if such triangle commutes the thesis follows. Then, make the* <u>upper</u> *square an* $F(A, -)$*-algebra morphism. Similarly, make the* <u>largest</u> *square an* $F(A, -)$*-algebra morphisms. Guess the last morphism* $F(A, X_B) \to F(A, X_C)$ *so that the* <u>lower</u> *square could be an* $F(A, -)$*-algebra morphism.*

   (b) *Prove that the lower square is indeed an* $F(A, -)$*-algebra morphism. For this, recall the commuting property seen at step 1 above for* X_h *and its* $F(B, -)$*-algebra. (Warning: this is* $F(B, -)$*, not* $F(A, -)$*.)*

   (c) *Conclude by initiality of* (X_A, f_A)*.*