Type Theory – 2020-07-21

Exercise 1. *Describe the encoding of product and coproduct types in System F. To do so, provide the corresponding System F types, as well as the related translation of* π_i *and* ι_i*.*

Exercise 2. *Replace the placeholders below so to make the following typing judgments correct according to the Calculus of Constructions.*

1) $\alpha : *,\ \beta : *,\ P : \alpha \to *,\ Q : \beta \to *,\ x : \prod_{a:\alpha} \prod_{f:\alpha\to\beta} Pa \to Q(fa),$
$y : \alpha,\ z : \beta,\ w : Py \vdash \boxed{?} : Qz$
2) $\alpha : *,\ P : \alpha \to *,\ w : \prod_{\beta:*} \prod_{a:\alpha}(Pa \to \beta) \to \beta\ \vdash \boxed{?} : \prod_{x:\alpha} Px$

Exercise 3. *Consider the following typing judgment in the simply-typed* λ *calculus. Below,* τ, σ *are basic types.*

$$x : \tau,\ y : \sigma \vdash (\lambda z : \tau.\ (\lambda w : \sigma.\ z)\ y)\ x : \tau \qquad (A)$$

1. *Reduce judgment* A *to normal form. Let* B *be the resulting judgment.*
2. *Show how to interpret* A *and* B *in a cartesian closed category* $\mathcal{C}$*, so obtaining two morphisms* a *and* b*.*
3. *Verify that* $a = b$*.*

Exercise 4. *In a parametric model of System F extended with* $0, 1, +, \times$*, consider the interpretation of the following type. Below,* τ, σ *are fixed types.*

$$\forall \alpha.\ (\sigma \to \alpha) \to (\tau \to \alpha) \to (\alpha \times \tau)$$

Construct a type γ *which is isomorphic to the above one but does not involve any quantifier* $\forall$ *in its definition. Justify your answer.*

Exercise 5. *Consider a locally small category* $\mathcal{C}$*, and let* $\mathcal{C}^2$ *be the product category* $\mathcal{C} \times \mathcal{C}$*. Let* $\Delta : \mathcal{C} \to \mathcal{C}^2$ *be the diagonal functor*

$$\Delta(X) = (X, X) \qquad \Delta(f) = (f, f)$$

Given three objects A, B, P *of* $\mathcal{C}$*, assume there exists a natural isomorphism (with respect to* X*, an object of* $\mathcal{C}$*)*

$$\eta_X : \mathcal{C}^2(\Delta(X), (A, B)) \cong \mathcal{C}(X, P) \qquad (\textit{in Set})$$

1. *[10%] Exploiting the above, craft a morphism in* $\mathcal{C}^2(\Delta(P), (A, B))$ *using the "obvious" construction. Name such morphism* $p = (p_A, p_B)$*.*
2. *[35%] Rewriting the definition of product object in an equivalent way, prove that* P *is the product object of* A *and* B *with projections* p_A *and* p_B *if and only if*

$$\forall X \in |\mathcal{C}|.\ \forall f \in \mathcal{C}^2(\Delta(X), (A, B)).\ \exists! m \in \mathcal{C}(X, P).\ f = \Delta(m); p \qquad (*)$$

3. *[20%] Provide the naturality law satisfied by* η*, and simplify it to an equation about morphisms of* $\mathcal{C}$ *(instead of about functions between sets of morphisms).*
4. *[35%] Use the naturality law and the definition of* p *to prove* $(*)$*, hence proving that* P *is the product.*